OHSTPY-HEP-T-95-024

Introduction to Theories of Fermion Masses

Stuart Raby
Department of Physics, The Ohio State University, 174 W. 18th Ave.
Columbus, OH 43210

ABSTRACT

This paper is based on four lectures given at the Trieste Summer School 1994 on theories of fermion masses. The first two lectures introduce three mechanisms which have been used to construct models of fermion masses. We then discuss some recent applications of these ideas. In the last lecture we briefly review SO(10) and some predictive theories of fermion masses.

1. Introduction

The Standard Model[SM] provides an excellent description of Nature. Myriads of experimental tests have to date found no inconsistency.

Eighteen phenomenological parameters in the SM are necessary to fit all the low energy data[LED].* These parameters are not equally well known. $\alpha, \sin^2(\theta_W)$, m_e, m_μ, m_τ and M_Z are all known to better than 1% accuracy. On the other-hand, $m_c, m_b, |V_{us}|$ are known to between 1% and 5% accuracy, and $\alpha_s(M_Z), m_u, m_d$, $m_s, m_t, |V_{cb}|, \left|\frac{V_{ub}}{V_{cb}}\right|, m_{Higgs}$ and the Jarlskog invariant measure of CP violation J are not known to better than 10% accuracy. One of the main goals of the experimental high energy physics program in the next 5 to 10 years will be to reduce these uncertainties. In addition, theoretical advances in heavy quark physics and lattice gauge calculations will reduce the theoretical uncertainties inherent in these parameters. Already the theoretical uncertainties in the determination of $|V_{cb}|$ from inclusive B decays are thought to be as low as 5%[1]. Moreover, lattice calculations are providing additional determinations of $\alpha_s(M_Z)$ and heavy quark masses[2].

Accurate knowledge of these 18 parameters is important. They are clearly not a random set of numbers. There are distinct patterns which can, if we are fortunate, guide us towards a fundamental theory which predicts some (if not all) of these parameters. Conversely these 18 parameters are the LED which will test any such theory. Note, that 13 of these parameters are in the fermion sector. So, if we are to make progress, we must necessarily attack the problem of fermion masses.

---

*This assumes the minimal particle content. With no right-handed neutrinos and only Higgs doublets, the theory predicts $m_\nu \equiv 0$.

1.1. Spinor Notation

There are 15 degrees of freedom in one family of fermions. We can describe these states in terms of 15 Weyl spinor fields (each field annihilates a particle with given quantum numbers and creates the corresponding anti-particle). We use the notation

$$(\nu\ e) \quad \overline{e} \quad (u\ d) \quad \overline{u} \quad \overline{d}$$

for these 15 fields (where the up and down quark fields have an implicit color index). The above fields are all left-handed Weyl spinors satisfying the free field equation of motion (in momentum space)

$$(E + \vec{\sigma} \cdot \vec{p})\omega(\vec{p}) = 0$$

where $E = |\vec{p}|$ and $\vec{\sigma}$ are 2×2 Pauli spinors. Rewriting this equation we find

$$\frac{\vec{\sigma} \cdot \vec{p}}{E}\omega(\vec{p}) = -\omega(\vec{p}) \equiv 2h\omega(\vec{p})$$

where $h = -1/2$ is the helicity of the state. Thus the states are left-handed ,i.e. their spin is anti-alligned with their momentum. We obtain a more compact notation by defining the Lorentz covariant spinor

$$\overline{\sigma}_\mu \equiv (1, \vec{\sigma}).$$

We then have $\not{P} \equiv P^\mu \overline{\sigma}_\mu = E + \vec{\sigma} \cdot \vec{p}$.

Note if $\omega(\vec{p})$ is a left-handed spinor, then $i\sigma_2\omega^*(\vec{p})$ satisfying

$$\frac{\vec{\sigma} \cdot \vec{p}}{E}(i\sigma_2\omega^*(\vec{p})) = +(i\sigma_2\omega^*(\vec{p}))$$

is right-handed.

Given the above notation, we can verify that we have accounted for all the degrees of freedom in one family. The field ν annihilates a left-handed neutrino and creates a right-handed anti-neutrino, while ν^* creates a left-handed neutrino and annihilates a right-handed anti-neutrino. The CP conjugate of ν is $\nu_{CP} = i\sigma_2\nu^*$.

For the electron we need two fields: e annihilates a left-handed electron and creates a right-handed anti-electron, while $\overline{e}$ annihilates a left-handed anti-electron and creates a right-handed electron. Whereas for the neutrino, only the combined operation CP can be defined; for the electron we can define the parity operation P such that $e_P = i\sigma_2\overline{e}^*$.

It is often useful when calculating Feynman amplitudes to use Dirac notation. We can always define a Dirac field in terms of two independent Weyl fields. For the electron we have

$$\Psi_e = \begin{pmatrix} e \\ i\sigma_2\overline{e}^* \end{pmatrix}.$$

In this basis, the Dirac gamma matrices are given by

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \overline{\sigma}_\mu & 0 \end{pmatrix}, \; \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

with $\sigma_\mu = (1, -\vec{\sigma})$. With this notation the left and right projectors are given by $P_{L(R)} = \frac{1-(+)\gamma_5}{2}$.

Lorentz scalars may be formed in the usual way. For example, a kinetic term for neutrinos is given by $\overline{\nu}_L \partial\!\!\!/ \nu_L = \nu^* \overline{\sigma}_\mu \partial^\mu \nu$. A majorana neutrino mass can be written as $\overline{\nu}_L \nu_R^C + h.c. = \nu\nu + h.c.$ where $\nu\nu \equiv \nu i \sigma_2 \nu$.

In the next section we use this formalism when discussing the fermionic sector of the Standard Model.

1.2. Standard Model[SM]

Consider the Yukawa sector of the SM. We have

$$\mathcal{L}_y = \mathcal{U}^{ij} \overline{u}_i^0 h Q_j^0 + \mathcal{D}^{ij} \overline{d}_i^0 \overline{h} Q_j^0 + \mathcal{E}^{ij} \overline{e}_i^0 \overline{h} L_j^0$$

The indices i,j= 1,2,3 label the three fermion families; $\mathcal{U}, \mathcal{D}, \mathcal{E}$ are complex 3×3 Yukawa matrices and $h, \overline{h}$ are Higgs doublets. In the minimal SM there is only one Higgs doublet and $\overline{h} \equiv i\tau_2 h^*$. However in any supersymmetric[SUSY] theory there are necessarily two independent Higgs doublets, so we will continue to refer to a theory with two independent Higgs doublets. The quark and lepton states are defined in terms of left-handed Weyl spinors and the superscript "0" refers to the so-called weak basis in which weak interactions are diagonal. In Table 1, we explicitly define the electroweak charge assignments of all the states.

We also define the vacuum expectation values[vev] of the Higgs fields in the conventional way

$$\langle h^0 \rangle = \frac{v_u}{\sqrt{2}}, \; \langle \overline{h}^0 \rangle = \frac{v_d}{\sqrt{2}}$$

with $v = \sqrt{v_u^2 + v_d^2} = 246 GeV$ as given by the tree relation $\frac{G_F}{\sqrt{2}} = \frac{g_2^2}{8 M_W^2} = \frac{1}{4v^2}$ or $v = (2\sqrt{2} G_F)^{-1/2}$. We also define the ratio of Higgs vevs $tan\beta \equiv v_u / v_d$. Thus fermion masses are given by

$$\begin{aligned} m_u \equiv & \; \mathcal{U} sin\beta \tfrac{v}{\sqrt{2}} \\ m_d \equiv & \; \mathcal{D} cos\beta \tfrac{v}{\sqrt{2}} \\ m_e \equiv & \; \mathcal{E} cos\beta \tfrac{v}{\sqrt{2}} \end{aligned}$$

In the weak basis, fermion mass matrices are non-diagonal complex 3×3 matrices. Note that CP invariance of the SM Lagrangian requires $m_a = m_a^*$ for a = u, d, e. Thus a non-removable phase in the fermion Yukawa matrices violates CP.

We can always diagonalize the mass matrices with the bi-unitary transformation

$$m_a^{Diag.} = \overline{V}_a m_a V_a^\dagger.$$

Table 1. Electroweak Charge Assignments.

State	Y – weak hypercharge	$SU(2)_L$
$Q^0 = \begin{pmatrix} u^0 \\ d^0 \end{pmatrix}$	1/3	doublet
$\overline{u}^0$	-4/3	singlet
$\overline{d}^0$	2/3	singlet
$L^0 = \begin{pmatrix} \nu^0 \\ e^0 \end{pmatrix}$	-1	doublet
$\overline{e}^0$	2	singlet
$h = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}$	1	doublet
$\overline{h} = \begin{pmatrix} \overline{h}^0 \\ \overline{h}^- \end{pmatrix}$	-1	doublet

The mass eigenstates are given by

$$u^0 \equiv V_u^\dagger u, \quad \overline{u}^0 \equiv \overline{u}\overline{V}_u$$

and similarly for down quarks and charged leptons. *There are 9 real (and by convention positive) mass parameters* given by $m_u,\ m_c,\ m_t,\ m_d,\ m_s,\ m_b,\ m_e,\ m_\mu,\ m_\tau$.

In the quark sector, the charged W interactions are given by the term $W^\mu (u^0)^* \overline{\sigma}_\mu d^0$ (in Dirac notation $\equiv W^\mu \overline{u}_L^0 \gamma_\mu d_L^0$) which in the mass eigenstate basis becomes

$$W^\mu \ u^* V_{CKM} \overline{\sigma}_\mu d.$$

The Cabbibo, Kobayashi, Maskawa matrix V_{CKM} is explicitly given by the expression

$$V_{CKM} \quad \equiv (V_u V_d^\dagger) \quad = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

Note, $V_{CKM}^\dagger V_{CKM} = 1$. Using the freedom to arbitrarily redefine the phases of all the fermions, the *CKM matrix can be expressed in terms of 4 real parameters (3 real angles and a CP violating phase). There are thus a total of 13 parameters in the fermion sector of the SM.* Note, since neutrinos are massless, we can always define a basis such that $\nu^0 \equiv V_e^\dagger \nu$. Thus there are no observable weak mixing matrices in the lepton sector of the theory.

1.3. Summary of Observable Fermionic Parameters

There is a hierarchy of fermion masses.

$$\begin{array}{ccccc} \tau(1777MeV) & > & \mu(105.6MeV) & > & e(.511MeV) \\ b(4.25 \pm .1GeV) & > & s(150 \pm 30MeV) & > & d(\sim 7MeV) \\ t(174 \pm 17GeV) & > & c(1.27 \pm 0.05GeV) & > & u(\sim 5MeV) \end{array}$$

There is a hierarchy of weak mixing angles as seen in the Wolfenstein parametrization of the CKM matrix.

$$V_{CKM} \approx \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho+i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho+i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

The parameters $\lambda \equiv |V_{us}| \approx .221$ and $A, \rho, \eta \sim 1$. Note $|V_{cb}| = A\lambda^2$ and $\left|\frac{V_{ub}}{V_{cb}}\right| = \lambda|\rho+i\eta|$. In this parametrization of the CKM matrix, η is the CP violating parameter. However this assignment depends explicitly on the particular phase convention chosen. A rephase invariant or convention independent CP violating parameter is given by the Jarlskog parameter J where

$$J \equiv Im(V_{ud}\, V_{ub}^*\, V_{tb}\, V_{td}^*).$$

There is a clear pattern of fermion masses and mixing angles. We would like to understand the origin of this pattern. But no one relation between parameters can provide that understanding. It can only come through a quantitative description of the whole pattern.

2. Renormalizability and Symmetry

The 18 phenomenological parameters of the SM are arbitrary independent renomalized parameters in the SM Lagrangian. Thus since they are arbitrary, *within the context of the SM they cannot be understood*. They are merely fit to the data. The problem of understanding these parameters is however even worse than you might think. In the fermionic sector of the theory there are 13 parameters. Consider however a single charge sector of fermions. For example, the complex 3×3 up quark matrix $\mathcal{U}$ has by itself 18 real arbitrary parameters. Thus in the fermionic sector there are in principle many more parameters than there are observables. This often leads to much confusion. In any fundamental theory of fermion masses, we would like to determine the Yukawa matrices $\mathcal{U}$, $\mathcal{D}$, $\mathcal{E}$. But only 13 combinations of the 54 parameters in these matrices are observable. In order to understand the pattern of fermion masses, it is necessary to reduce the number of arbitrary parameters in the Yukawa matrices from 54 to a number which is less than 13.

The key ingredients which may allow us to make some progress in this direction are renomalizable field theories and symmetry. In a renormalizable field theory there

are only a finite number of counterterms necessary to define the theory. For example in QED, we have the renormalized Lagrangian given by

$$\mathcal{L} = Z_2 \overline{\Psi} \partial\!\!\!/ \Psi + Z_1 e \overline{\Psi} A\!\!\!/ \Psi - Z_3 \frac{1}{4} F_{\mu\nu}^2 - Z_m m \overline{\Psi} \Psi.$$

In this case the electron charge and mass are arbitrary parameters. However if we can introduce enough symmetry into a theory such that there are more observable parameters than there are counterterms, we can in principle obtain predictable relations among these parameters.

In the following, we will discuss three mechanisms which have been used in the past for obtaining relations between fermion masses and mixing angles. We will then discuss more recent realizations using these 3 tools of the trade.

2.1. Tools of the Trade

Before describing each mechanism in detail, let me give a brief description of the seminal ideas involved. We will broadly classify the 3 mechanisms as *radiative*, *textures* and *effective operator* relations.

- *Radiative* In this example, we calculate the electron mass as a radiative correction proportional to the muon mass. We show that the gauge symmetry of the theory allows only one Yukawa coupling for both μ and e. In addition, as a consequence of a missing vacuum expectation value[vev], the muon obtains mass at tree level while the electron remains massless. At one loop we then find $m_e \sim \alpha m_\mu$.

- *Textures* We use both gauge and discrete family symmetries to define the most general Yukawa matrix for a pair of quarks which is symmetric and has a certain number of zero elements, thereby reducing the number of fundamental parameters. We thus obtain tree level relations among quark masses and mixing angles. Note since experimentally $m_d/m_s \sim 1/20 >> \alpha$, it would not be possible to obtain all mass ratios radiatively.

- *Effective Operators* We use U(1) symmetries with light fermions coupled to heavy fermions with mirror partners. When integrating the heavy fermions out of the theory we generate effective higher dimension fermion mass operators which explain the fermion mass hierarchy.

2.2. Radiative mechanism

[Weinberg, 1972; Georgi and Glashow, 1973][3] In a seminal paper, coming shortly after the proof of the renormalizability of spontaneously broken non-abelian gauge theories[4], Weinberg emphasized the advantages of renormalizable field theories for

obtaining calculable fermion masses. In a simple example he showed that the electron mass can be generated radiatively from the muon mass. There was a critical flaw in his example which was later pointed out and corrected by Georgi and Glashow.

Consider a theory describing just two families of leptons. The electroweak gauge group is extended to $G_W = SU(3)_1 \times SU(3)_2$ which allows only one Yukawa coupling λ for both μ and e. In addition the theory has a discrete parity invariance which interchanges the states transforming under the two SU(3)s, i.e. $1 \leftrightarrow 2$ which among other things allows only one gauge coupling constant, g. The fermions are represented by

$$\Psi_1 \equiv \begin{pmatrix} \overline{\mu} \\ \nu_e \\ e \end{pmatrix}, \quad \Psi_2 \equiv \begin{pmatrix} \overline{e} \\ \nu_\mu \\ \mu \end{pmatrix}$$

which are in the (3,1), (1,3) representation of G_W, respectively. The minimal Higgs content $\Phi_{ab} = (\overline{3}, \overline{3}),\ a, b = 1, 2, 3$ contains two Higgs doublets when looked at in terms of the $SU(2)_L \times U(1)_Y$ subgroup of G_W. The $SU(2)_L \times U(1)_Y$ subgroup is explicitly defined by the generators $T_i = t^1_i + t^2_i,\ i = 1, 2, 3$ and $Y = -\sqrt{12}(t^1_8 + t^2_8)$ with

$$t_8 \equiv \frac{1}{\sqrt{12}} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The only renormalizable Yukawa coupling is given by

$$\lambda \Psi^a_1 \Phi_{ab} \Psi^b_2.$$

As a result, the μ, e masses are given in terms of the expressions $m_\mu = \lambda v_\mu,\ m_e = \lambda v_e$ where $v_\mu = \langle \Phi_{13} \rangle,\ v_e = \langle \Phi_{31} \rangle$ are the two vevs of Φ which break $SU(2)_L \times U(1)$ to $U(1)_{EM}$. The most general renormalizable potential for Φ is defined such that, for a finite range of parameters, the minimum energy state has $v_\mu \neq 0, v_e = 0$. Thus the electron is massless at tree level. However there is no symmetry which can protect the electron from obtaining a mass radiatively since the chiral symmetries of e and μ are united by the gauge group G_W. In fig. 1 we show the Feynman diagram which contributes to the electron mass.

The problem with this model, discovered by Georgi and Glashow, is evident from the Feynman diagram of fig. 2. This diagram is obtained by closing the external fermion line in fig. 1. This diagram is logarithmically divergent. It in fact generates the local dimension 4 operator

$$\Phi^*_{31} \Phi_{13} X X^*.$$

Such a term must be in the Lagrangian since there is clearly no symmetry which prevents it and it is a dimension 4 operator which requires a fundamental parameter to renormalize. This term has the nasty effect of driving $v_e = \langle \Phi_{31} \rangle \neq 0$. In order to solve this problem and have a renomalizable scalar potential such that $v_e = 0$ "naturally", Georgi and Glashow proposed to enlarge the gauge symmetry G_W further. The details are not important. It is important to recognize however that the problem for

Weinberg's example is that the most general renormalizable potential for Φ did not satisfy the requirement that, for a finite range of parameters, the minimum energy state has $v_e = 0$.

2.3. Textures

[Weinberg; Wilczek and Zee; Fritzsch, 1977][5] In 1968, several people made the observation of a simple empirical relation between the Cabibbo angle and the down and strange quark mass ratio given by[6]

$$tan\Theta_c = \sqrt{\frac{m_d}{m_s}} \approx \frac{f_\pi m_\pi}{f_K m_K}.$$

It was not until 9 years later that a possible explanation of this relation was proposed.[5]

Before we discuss the explanation, let's consider the general problem. The up and down quark mass terms, defined in the weak eigenstate basis, are given by

$$\delta\mathcal{L} = \begin{pmatrix} \overline{u} & \overline{c} \end{pmatrix} m_u \begin{pmatrix} u \\ c \end{pmatrix} + \begin{pmatrix} \overline{d} & \overline{s} \end{pmatrix} m_d \begin{pmatrix} d \\ s \end{pmatrix}$$

where in general the up and down mass matrices are given by

$$m_u = \begin{pmatrix} \tilde{C} & \tilde{B}' \\ \tilde{B} & \tilde{A} \end{pmatrix},$$

$$m_d = \begin{pmatrix} C & B' \\ B & A \end{pmatrix}.$$

$A, B, B', C, \tilde{A}, \tilde{B}, \tilde{B}', \tilde{C}$ are in general arbitrary complex parameters. Note however that not all the phases are physical. We can redefine the phases of the fields $u, \overline{u}$, $c, \overline{c}$, $d, \overline{d}$, $s, \overline{s}$ and remove 5 of these phases *without introducing any new phases anywhere else in the Lagrangian.* For example, redefine the phase of $\overline{s}$ to make B real, then redefine the phase of s to make A real. Note that we must also redefine the phase of c by the same amount as s so that we don't introduce a new phase in the W-c-s vertex. Next redefine the phases of $\overline{d}, \overline{c}, \overline{u}$ making B' and $\tilde{A}$ real and $arg\tilde{B} = -arg\tilde{B}'$. We now see that the mass eigenstates and mixing angles in the up (down) sector depend on 6 (5) parameters for a total of 11 parameters. However, how many observables are there? There are 4 quark masses and one electroweak mixing angle or a total of 5 parameters. We certainly have enough arbitrary parameters to fit these 5 observables, but we are not able to make any predictions. In order to make predictions we must reduce the number of arbitrary parameters. In order to reduce the number of fundamental parameters we need to introduce symmetries. In the paper by Fritzsch (see ([5])) it was shown that by

- extending the electroweak gauge symmetry to $SU(2)_L \otimes SU(2)_R \otimes U(1)$, and

- demanding Parity, CP and an additional discrete symmetry

the number of arbitrary parameters in m_u, m_d can be reduced to 4. This allows for one prediction which relates masses and the one mixing angle. The discrete symmetry enforces $C = \tilde{C} = 0$ and parity requires the matrices be Hermitian. The resulting matrices have the form

$$m_u = \begin{pmatrix} 0 & |\tilde{B}| \\ |\tilde{B}| & |\tilde{A}| \end{pmatrix},$$

$$m_d = \begin{pmatrix} 0 & |B|e^{-i\gamma} \\ |B|e^{i\gamma} & |A| \end{pmatrix}.$$

Note without assuming CP (which requires m_u and m_d to be real) we remain with one phase and lose the prediction.

Let me now give Fritzsch's model in more detail. The model included 2 left-handed quark and 2 left-handed anti-quark doublets

$$Q_i = \begin{pmatrix} u \\ d \end{pmatrix}_i, \quad \overline{Q}_i = \begin{pmatrix} \overline{u} \\ \overline{d} \end{pmatrix}_i, \quad i = 1, 2$$

transforming in the $(2,1), (1,\overline{2})$ representations of $SU(2)_L \otimes SU(2)_R \otimes U(1)$ with equal and opposite U(1) charges. The index i is a generation label. Thus you should consider the quarks denoted by 2 as c and s quarks and those by 1 as u and d. In addition, the model has 2 scalar multiplets $\phi_{1(2)}$ in the $(\overline{2},\overline{2})$ representation. Without any additional symmetries the allowed scalar -quark -anti-quark couplings are given by

$$\delta\mathcal{L} = \lambda_{ij}\overline{Q}_i\phi_1 Q_j + \lambda'_{ij}\overline{Q}_i\phi_2 Q_j$$

with $\lambda_{ij}, \lambda'_{ij}$ arbitrary complex coupling constants. If we now imposed CP invariance on the Lagrangian, then λ_{ij} and λ'_{ij} are real. Under Parity

$$Q \leftrightarrow i\sigma_2\overline{Q}^*, \ \phi \leftrightarrow \sigma_2\phi^\dagger\sigma_2.$$

Imposing P on the Lagrangian requires

$$\lambda_{ij} = \lambda^*_{ji}, \ \lambda'_{ij} = (\lambda'_{ji})^*.$$

Finally, we define two additional discrete symmetries $\{P_1, P_2\}$ which act on the set of fields in the following way.

$$P_1 : \begin{array}{c} \{Q_1, \overline{Q}_1, \phi_1\} \to (-1) \times \{Q_1, \overline{Q}_1, \phi_1\} \\ \{Q_2, \overline{Q}_2, \phi_2\} \to (i) \times \{Q_2, \overline{Q}_2, \phi_2\} \end{array}$$

$$P_2 : \{Q_2, \overline{Q}_2, \phi_2\} \to (-1) \times \{Q_2, \overline{Q}_2, \phi_2\}.$$

I list below the only terms in $\delta\mathcal{L}$ allowed by P_2 –

$$\lambda'_{12}(\overline{Q}_2\phi_2 Q_1 + \overline{Q}_1\phi_2 Q_2) + \lambda_{22}(\overline{Q}_2\phi_1 Q_2) + \lambda_{11}(\overline{Q}_1\phi_1 Q_1).$$

If we now impose P_1 we are left with

$$\lambda'_{12}(\overline{Q}_2\phi_2 Q_1 + \overline{Q}_1\phi_2 Q_2) + \lambda_{22}(\overline{Q}_2\phi_1 Q_2).$$

The up and down quark mass matrices are now given by

$$m_u = \begin{pmatrix} 0 & \lambda'_{12}\langle\phi_2^u\rangle \\ \lambda'_{12}\langle\phi_2^u\rangle & \lambda_{22}\langle\phi_1^u\rangle \end{pmatrix} \equiv \begin{pmatrix} 0 & \tilde{B} \\ \tilde{B} & \tilde{A} \end{pmatrix}$$

and m_d is given by the same expression with $\phi^d_{1(2)}$ replacing $\phi^u_{1(2)}$ or

$$m_d \equiv \begin{pmatrix} 0 & B \\ B & A \end{pmatrix}.$$

Note $\phi^{u(d)}$ are the neutral components of the scalar ϕ which give mass to up (down) quarks. The weak vev v is given by $v = \sqrt{(\phi_1^u)^2 + (\phi_2^u)^2 + (\phi_1^d)^2 + (\phi_2^d)^2}$.

We can now obtain the successful relation

$$tan\Theta_c \approx (\sqrt{\frac{m_u}{m_c}} - \sqrt{\frac{m_d}{m_s}}).$$

2.4. Effective Operators

[Froggatt and Nielsen, 1979][7] In the previous mechanism, the small mass ratios $m_u/m_c(m_d/m_s)$ are given in terms of arbitrary ratios $\tilde{B}^2/\tilde{A}^2(B^2/A^2)$. But we have no understanding of why $\tilde{B} << \tilde{A}$, etc. Froggatt and Nielsen tried to provide this explanation.

Consider the SM with an additional global U(1) symmetry denoted by Q. The quantum numbers of , for example, up-type quarks under Q are given by $\overline{q}_i \equiv q(\overline{u}_i), \;\; q_i \equiv q(u_i)$ and we take $q(Higgs) \equiv 0$. We assume that Q is spontaneously broken and that the symmetry breaking is communicated to quarks by the insertion of a tadpole with magnitude $\epsilon < 1$ and charge -1. It is then assumed that $\overline{q}_3 = q_3 = 0$ with $\overline{q}_i, \;\; q_i$ non-vanishing such that the Yukawa term $\overline{u}_3 h u_3$ is the only Q invariant term without a symmetry breaking insertion. The term $\overline{u}_i h u_j$ has Q charge $\overline{q}_i + q_j$ and needs the insertion $\epsilon^{\overline{q}_i + q_j}$ to be invariant (see fig. 3). These effective higher dimension operator terms are thus suppressed with respect to the direct dimension four Yukawa coupling.

What is the origin of the small parameter ϵ? Consider the graphs of fig. 4 which describes a simple two family quark model. We have introduced two new scalars ϕ^0, ϕ^{-1}, singlets under the electroweak symmetry with Q charge denoted by the superscript and the left-right symmetric up-type quarks $U^{\pm 1}, \overline{U}^{\pm 1}$, members of an $SU(2)_L$ doublet, anti-doublet, respectively. Both ϕ^0, ϕ^{-1} are assumed to get non-vanishing vevs satisfying $\phi^0 > \phi^{-1} >> M_Z$. As a result, the new up-type quarks

are heavy with mass of order $\langle\phi^0\rangle$. Due to the expectation values of the weak Higgs h^0 and the new scalar ϕ^{-1}, these heavy quarks mix with the light quarks. Fig. 4 represents this mixing. These graphs generate off-diagonal mixing in the fermion mass matrices between the light up quarks of the second and third family. We have $\epsilon(\overline{u}_2^1 h u_3^0 + \overline{u}_3^0 h u_2^1)$ as the lowest order mixing obtained in a power series expansion in the small parameter $\epsilon = \frac{\langle\phi^{-1}\rangle}{\langle\phi^0\rangle}$.

The procedure of reading the low energy mixing term off of the diagram of fig. 4 is equivalent to the procedure of diagonalizing the fermion mass matrix, ignoring the weak vev of h. For example, consider the mass terms represented as vertices in fig. 4 for the state $\overline{U}^{+1}$. We have

$$\overline{U}^{+1}(\langle\phi^0\rangle U^{-1} + \langle\phi^{-1}\rangle u_3^0) \approx \langle\phi^0\rangle\overline{U}^{+1}(U^{-1} + \epsilon u_3^0).$$

Define the massive state $u_M \approx U^{-1} + \epsilon u_3^0$ and the orthogonal massless state is $u_3'^0 \approx -\epsilon U^{-1} + u_3^0$. At energies much below $\langle\phi^0\rangle$ and greater than $\langle h\rangle$ we can define an effective theory by integrating out the states with mass of order $\langle\phi^0\rangle$. In this effective theory, the vertex $\overline{u}_2^1 h U^{-1}$ becomes $-\epsilon\overline{u}_2^1 h u_3'^0$ which is obtained by using the relation $U^{-1} \approx -\epsilon u_3'^0 + u_M$. Of course, the exact effective dimension 4 Yukawa coupling (which contains an expansion in ϵ) is obtained by using the exact expressions for the massive and massless eigenstates.

In this mechanism the extra global symmetry Q controls the textures of effective mass operators in fig. 3.

3. Theories of Fermion Masses - Survey (1979 - 1994)

In the last 15 years, there have been many papers on fermion masses. Most of these papers, if not all of them, have been applications of one or more of the mechanisms or tools for fermion masses we discussed in the previous section. In this section I will consider a few representative examples of papers in the literature. I make no claim that these examples are all inclusive.

3.1. Radiative mechanism

The extended technicolor theory of fermion masses assumes that the light quarks and leptons receive their mass via a radiative mechanism from new heavy technifermions. The technifermion mass, on the otherhand, results from a chiral symmetry breaking condensate due to new strong technicolor interactions[see Dimopoulos and Susskind, Eichten and Lane][8]. These models are notoriously non predictive as a result of the strong interactions which are needed for chiral symmetry breaking. One can at best obtain order of magnitude estimates for quark masses given by formulae such as $m_q \sim \frac{\langle\overline{T}T\rangle}{\Lambda_{ETC}^2}$ where $\langle\overline{T}T\rangle$ is the technifermion condensate and Λ_{ETC} is the ETC breaking scale.

In recent models, people have attempted to get fermion masses in SUSY theories by feeding masses from the squark and slepton sector into the quark and lepton

sectors[9]. For example, in the graph of fig. 5 the down quark gets mass from a soft SUSY breaking bottom squark mass squared given by Am_b. This leads to a down quark mass

$$m_d \approx \frac{\alpha_s}{2\pi}\left(\frac{\delta\tilde{m}^2}{\tilde{m}^2}\right)^2\left(\frac{Am_{\tilde{g}}}{\tilde{m}^2}\right)m_b$$

where $\delta\tilde{m}^2$ is a measure of the bottom and down squark mixing in the quark-squark basis which diagonalizes quark masses at tree level. Since such a theory replaces the arbitrary Yukawa parameters in the fermion sector by new arbitrary mass parameters in the scalar sector, it is not clear that one can really make progress using this paradigm.

3.2. Textures and Discrete Symmetries

Fritzsch generalized his theory of the Cabibbo angle to a complete 6 quark model[10]. This model has 6 real magnitudes and 2 phases or eight parameters to fit 10 observables (six quark masses and 4 CKM angles). There are thus 2 predictions. One of these predictions, as shown by Gilman and Nir[10], is that the top quark is necessarily light, i.e. $m_t < 96GeV$. Hence the Fritzsch texture is now ruled out by experiments at Fermilab.

3.3. Effective Operators

A SUSY version of the Froggatt and Nielsen mechanism has recently been studied in the literature[11] within the context of the SM gauge group. In these models the fermion mass matrices have the form

$$m_{ij} = \epsilon^{q(\overline{f}_i)+q(f_j)}$$

where $f = u, d, e$. This paradigm can "naturally" explain the zeros in mass matrices and certain order of magnitude ratios of non-vanishing elements, but unfortunately it has no power to predict testable fermion mass relations. The proof of this paradigm would be found in the existence of new states with mass above the weak scale responsible for the effective operators.

All of the examples discussed so far have the following features in common:

1. They are all relations defined just above the weak scale; as a consequence they all require new physics (new particles and/or gauge symmetries) just above experimental observation.

2. They all (except for the Fritzsch Ansatz) give only a qualitative description of fermion masses; thus there are no testable predictions for fermion masses and mixing angles.

3. Quark and lepton masses are unrelated.

An important question is what is the scale of new physics; the scale at which new symmetries and particles appear. If this new scale is just above the weak scale then we must worry about possible new flavor changing neutral current[FCNC] interactions. In radiative mechanisms, loop diagrams can contribute to new FCNC interactions (for example see fig. 6). In this case the effective FCNC interactions are of order

$$\delta\mathcal{L} \sim \alpha_W^2 \left(\frac{\delta\tilde{m}^2}{\tilde{m}^2}\right) \frac{1}{\tilde{m}^2}(s^* d)^2.$$

They are proportional to squark mixing mass terms and can be suppressed by increasing the overall squark mass scale. In the texture mechanism, the new states required to incorporate the necessary discrete and gauge symmetries which make texture zeros "natural" will contribute to FCNC interactions. Finally in the Froggatt-Nielsen mechanism, the new heavy fermions and scalars can also contribute to FCNC interactions. In all cases, one must compare the new FCNC interactions with experiment and place bounds on the scale of new physics. Generically, these bounds will force the new physics scale to be in the $(1 - 10^3) TeV$ range depending in detail on the specific process considered.

4. SUSY GUTs

We would like to obtain models *more predictive* than our previous examples. In order to do this we need *more symmetry.* We can gain a lot of predictive power by relating quark and lepton masses. Of course this requires some sort of grand unification symmetry[12].

In the rest of these lectures I will consider the consequences of SUSY Grand Unified Theories [GUTs][13,14]. The main reason is that they already make one prediction which agrees remarkably well with low energy data[15]. Using the measured values of α and $sin^2\theta_W$, and assuming reasonable threshold corrections at the weak and GUT scales, Langacker and Polonsky[16] obtain the prediction for $\alpha_s(M_Z)$ in fig. 7. They also plot the experimental measurements of $\alpha_s(M_Z)$ and you can see that the two are in remarkable agreement. Note that the minimal non-SUSY GUT gives a value for $\alpha_s(M_Z) \sim 0.07$ which is several standard deviations away from the observations.

Let us now consider the first predictions from GUTs for fermion masses. In order to do this we will give a brief introduction to $SU(5)$[12]. The quarks and leptons in one family of fermions fit into two irreducible representations of $SU(5)$: $10_{ij} = -10_{ji}$, $\bar{5}^i = \epsilon^{ijklm}\bar{5}_{jklm}$ where $i, j, k, l, m = 1 - 5$ are $SU(5)$ indices. In the fundamental 5 dimensional representation of $SU(5)$ the adjoint is represented by 5×5 traceless hermitian matrices. We can consider the indices from $1 - 3$ as being color indices acted on by the $SU(3)_{color}$ subgroup of $SU(5)$ and the indices $4, 5$ as weak

$SU(2)_L$ indices. Hypercharge is represented by the matrix

$$Y = -2\sqrt{\frac{5}{3}} Y_5 \text{ and } Y_5 \equiv \frac{1}{\sqrt{60}} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}$$

satisfying $Tr Y_5^2 = 1/2$. From this embedding of the SM into $SU(5)$ we can check that the states fit into the 10 and $\overline{5}$ as follows:

$$10 = \begin{pmatrix} \overline{u} & Q \\ & \overline{e} \end{pmatrix}, \quad \overline{5} = \begin{pmatrix} \overline{d} \\ L \end{pmatrix}.$$

The two Higgs doublets fit into a $5(\equiv H)$ and $\overline{5}(\equiv \overline{H})$. Similarly H and $\overline{H}$ can be decomposed into weak doublets and color triplets under the SM symmetry. We have

$$\overline{H} = \begin{pmatrix} \overline{t} \\ \overline{h} \end{pmatrix}, \quad H = \begin{pmatrix} t \\ h \end{pmatrix}$$

with $t(h)$ denoting triplet(doublet) states.

Up and down quark Yukawa couplings at M_{GUT} are given in terms of the operators

$$\lambda_u H_i 10_{jk} 10_{lm} \epsilon^{ijklm} + \lambda_d \overline{H}^i 10_{ij} \overline{5}^j.$$

When written in terms of quark and lepton states we obtain the Yukawa couplings to the Higgs doublets

$$\lambda_u \overline{u} h Q + \lambda_d (\overline{d}\overline{h} Q + \overline{e}\overline{h} L).$$

We see that $SU(5)$ relates the Yukawa couplings of down quarks and charged leptons, i.e. $\lambda_d = \lambda_e$ at the GUT scale. <u>Assuming this relation holds for all 3 families</u>, we have[17] $\lambda_b = \lambda_\tau, \;\; \lambda_s = \lambda_\mu, \;\; \lambda_d = \lambda_e$ at M_{GUT}.

To compare with experiment we must use the renormalization group[RG] equations to run these relations (valid at M_{GUT}) to the weak scale. The first relation gives a prediction for the b-τ ratio which is in good agreement with low energy data. Note, for heavy top quarks we must now use the analysis which includes the third generation Yukawa couplings[18]. We will discuss these results shortly. The next two relations can be used to derive the relation: $\frac{\lambda_s}{\lambda_d} = \frac{\lambda_\mu}{\lambda_e}$ at M_{GUT}. However at one loop the two ratios are to a good approximation RG invariants. Thus the relation is valid at any scale $\mu < M_{GUT}$. This leads to the *bad* prediction

$$\frac{m_s}{m_d} = \frac{m_\mu}{m_e}$$

for running masses evaluated at 1 GeV. It is a bad prediction since experimentally the left hand side is ~ 20 while the rhs is ~ 200.

An ingenious method to fix this bad relation was proposed by Georgi and Jarlskog[19]. They show how to use $SU(5)$ Clebschs in a novel texture for fermion Yukawa matrices to keep the good relation $\lambda_b = \lambda_\tau$, and replace the bad relation above by the good relation

$$\frac{m_s}{m_d} = \frac{1}{9}\frac{m_\mu}{m_e}.$$

4.1. Georgi-Jarlskog Texture

Georgi and Jarlskog found an interesting texture which resolved the problem of light fermion masses. They also constructed a grand unified theory with 3 families of quarks and leptons, the necessary Higgs and with sufficient symmetry so that the theory was "natural."

The fermion Yukawa matrices have the form

$$\mathcal{U} = \begin{pmatrix} 0 & C & 0 \\ C & 0 & B \\ 0 & B & A \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} 0 & F & 0 \\ F' & E & 0 \\ 0 & 0 & D \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} 0 & F' & 0 \\ F & -3E & 0 \\ 0 & 0 & D \end{pmatrix}$$

where A, B, C, D, E, F, F' are in general arbitrary complex parameters. The $SU(5)$ version of the theory contains in addition to the Higgs multiplets, $\overline{H} = \overline{5}, \; H = 5$ discussed previously, a $\overline{45}$. The Yukawa Lagrangian is given by

$$\overline{H}(F' 10_1 \overline{5}_2 + F 10_2 \overline{5}_1) + D\overline{H} 10_3 \overline{5}_3 + E\overline{45} 10_2 \overline{5}_2$$
$$+ CH 10_1 10_2 + BH 10_2 10_3 + AH 10_3 10_3.$$

Note if we diagonalize the down and charged lepton matrices in the 2×2 subspace of the two light generations we find the relations $\lambda_s \approx \frac{1}{3}\lambda_\mu, \;\; \lambda_d \approx 3\lambda_e$ resulting from the Clebsch factor of 3. This factor of 3 is very natural in any GUT since it just results from the fact that there are three quark states for every lepton state. After RG running from M_{GUT} to 1 GeV we obtain the good mass relations $m_s \approx \frac{4}{3}m_\mu, \; m_d \approx 12 m_e$.

Note, the up mass matrix is necessarily symmetric but within $SU(5)$ the down matrix is not. It was shown by Georgi and Nanopoulos[20] that by extending the gauge symmetry to $SO(10)$ the down matrix will also be symmetric. In this case a 126 dimensional representation is needed to obtain the Clebsch of 3. A complete SO(10) version of the theory was first given in a paper by Harvey, Ramond and Reiss[20].

Since SUSY GUTs seem to work so nicely for gauge coupling unification, it is natural to wonder whether a SUSY version of the Georgi-Jarlskog ansatz gives reasonable predictions for fermion masses and mixing angles. Dimopoulos, Hall and I showed that the predictions for fermion masses and mixing angles worked very well[21,22]. Using the freedom to redefine the phases of fermions, we showed that there were just 7 arbitrary parameters in the Yukawa matrices; a standard form is given by

$$\mathcal{U} = \begin{pmatrix} 0 & C & 0 \\ C & 0 & B \\ 0 & B & A \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} 0 & Fe^{i\phi} & 0 \\ Fe^{-i\phi} & E & 0 \\ 0 & 0 & D \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} 0 & Fe^{i\phi} & 0 \\ Fe^{-i\phi} & -3E & 0 \\ 0 & 0 & D \end{pmatrix}$$

where now A, B, C, D, E, F, ϕ are the 7 real parameters. Including $tan\beta$ we have 8 real parameters in the fermion mass matrices. On the otherhand, there are 14 low energy observables, 9 charged fermion masses, 4 quark mixing angles and $tan\beta$; thus there are 6 predictions.

We used the best known low energy observables, $m_e, m_\mu, m_\tau, m_c, m_b, |V_{us}|, \frac{m_u}{m_d}$ as input to make predictions for $m_t, m_s, |V_{cb}|, m_d, \left|\frac{V_{ub}}{V_{cb}}\right|$ and the CP violating parameter J in terms of arbitrary values of $tan\beta$. The results were in good agreement with the low energy data. Fitting all the parameters simultaneously, Barger, Berger, Han and Zralek[22] showed this texture agreed with all the low energy data at 90% CL. Recently Babu and Mohapatra have found an interesting representation for the Georgi-Jarlskog texture in terms of an effective theory at M_{Planck}[23]. This is an SO(10) theory containing effective mass operators with dimension ≥ 4 which eliminates the need for the large 126 dimensional representation.

To conclude this review of the literature, other textures have recently been pursued. Different SO(10) SUSY GUT textures have been discussed[24]. Babu and Shafi have considered the SUSY version of Fritzsch (defined at a GUT scale) and showed that $m_t < 120GeV$[25]. Finally Ramond, Roberts and Ross have, in a bottom-up approach, classified all symmetric quark mass matrices within the minimal supersymmetric standard model[MSSM] with texture zeros at M_{GUT}[26]. They find 6 solutions which fit the data.

4.2. Renormalization Group Running

In this section I want to discuss the RG equations for the bottom, top and tau Yukawa couplings in SUSY GUTs. The RG equations described below neglect mixing among the different generations. For the light families one can neglect the effect of the Yukawa couplings in the beta functions on the right-hand-side of these equations. Specifying a typical Yukawa coupling by λ we define the quantity $Y \equiv \lambda^2/(4\pi)^2$. We also define $\tilde{\alpha}_i \equiv \alpha_i/(4\pi)$ and $t = ln\frac{M_G^2}{\mu^2}$. In terms of these parameters the RG equations are[18]:

$$\frac{dY_t}{dt} = Y_t(\frac{16}{3}\tilde{\alpha}_3 + 3\tilde{\alpha}_2 + \frac{13}{9}\tilde{\alpha}_1 - 6Y_t - Y_b - Y_{\nu_\tau}\theta),$$

$$\frac{dY_b}{dt} = Y_b(\frac{16}{3}\tilde{\alpha}_3 + 3\tilde{\alpha}_2 + \frac{7}{9}\tilde{\alpha}_1 - Y_t - 6Y_b - Y_\tau),$$

$$\frac{dY_\tau}{dt} = Y_\tau(3\tilde{\alpha}_2 + 3\tilde{\alpha}_1 - 3Y_b - 4Y_\tau - Y_{\nu_\tau}\theta),$$

and

$$\frac{dY_{\nu_\tau}}{dt} = Y_{\nu_\tau}(3\tilde{\alpha}_2 + \frac{3}{5}\tilde{\alpha}_1 - 3Y_t - 4Y_{\nu_\tau}\theta - Y_\tau).$$

We have included the RG equations for the tau neutrino assuming a Dirac mass term for the left handed tau neutrino coupled to a singlet state. The function θ is

zero for $t > ln\frac{M_G^2}{M^2}$ and one otherwise, where M is the majorana mass of the singlet neutrino. For $M \sim M_{GUT}$ the tau neutrino does not affect the running of the charged fermions.

For light quarks and leptons it is easy to see that the additional color interactions for quarks explains why the ratio of quark to lepton Yukawa couplings increases at low energy. For the bottom to tau mass ratio this increase is in fact too large if one begins with the unification assumption that $\lambda_b = \lambda_\tau$ at M_{GUT}. It was shown by Inoue et. al. and Ibañez and Lopez that a large top quark Yukawa coupling can decrease the ratio λ_b/λ_τ at low energies[18]![†] In fig. 8 we show this ratio as a function of the top quark mass valid for small $tan\beta$ or equivalently neglecting λ_b and λ_τ in the RG runnning. In fig. 9 we show the prediction for the top quark running mass as a function of $tan\beta$ assuming $b - \tau$ unification at M_{GUT}[15]. You see that the top quark is naturally heavy and can easily be in the range observed at Fermilab.

As an aside, it has been noted recently by several authors[27] that the tau neutrino can affect the RG equations significantly if its mass is in the few eV range making it a good candidate for a hot component to the dark matter in the universe. In this case $M \sim 10^{12} GeV$ and the tau neutrino becomes important. They noticed that the tau neutrino offsets the effect of the top quark Yukawa coupling to decrease the bottom to tau mass ratio. See for example the equation below.

$$\frac{d}{dt}\left(\frac{Y_b}{Y_\tau}\right) = \left(\frac{Y_b}{Y_\tau}\right)(\frac{16}{3}\tilde{\alpha}_3 - \frac{20}{9}\tilde{\alpha}_1 - (Y_t - Y_{\nu_\tau}\theta) - 3(Y_b - Y_\tau)).$$

In order to affect a significant decrease they show that the bottom quark Yukawa coupling, which also tends to drive the bottom to tau ratio down, must be significant, requiring values of $tan\beta$ larger than about 10.

Finally Ananthanarayan, Lazarides and Shafi[28] have studied the SO(10) GUT boundary conditions $\lambda_t = \lambda_b = \lambda_\tau$. They have demonstrated that these conditions are consistent with the low energy data. They necessarily require large values of $tan\beta \sim 50$. We will study this case in more detail, but first let me briefly discuss the group SO(10).

5. Introduction to SO(10) Group Theory

The defining representation is a ten dimensional vector denoted by $10_i, \quad i = 1, \cdots, 10$. SO(10) is defined by the set of real orthogonal transformations $O_{ij} : O^T O = 1$ such that $10'_i = O_{ij} 10_j$. Infinitesimal SO(10) rotations are given by $O = 1 + i\tilde{\omega}$ with $\tilde{\omega}^T = -\tilde{\omega}$. We can always express the 10×10 antisymmetric matrix $\tilde{\omega}$ in the canonical form $\tilde{\omega}_{ij} \equiv \omega_{ab}\Sigma^{ab}_{ij}$. ω_{ab} are 45 real infinitesimal parameters satisfying $\omega_{ab} = -\omega_{ba}$ and $\Sigma^{ab}_{ij} = i(\delta^a_i\delta^b_j - \delta^a_j\delta^b_i)$ are the 45 generators of SO(10) in the 10 dimensional representation. Note that the antisymmetric tensor product $(10 \times 10)_A \equiv 45$ is the adjoint representation.

[†]In a one Higgs model, however, the top quark Yukawa coupling tends to increase the ratio λ_b/λ_τ

The SO(10) generators satisfy the Lie algebra

$$[\Sigma^{ab},\ \Sigma^{cd}]_{ik} \equiv \Sigma^{ab}_{ij}\Sigma^{cd}_{jk} - \Sigma^{cd}_{ij}\Sigma^{ab}_{jk} = [\Sigma^{ad}_{ik}\delta_{bc} - \Sigma^{ac}_{ik}\delta_{bd} + \Sigma^{bc}_{ik}\delta_{ad} - \Sigma^{bd}_{ik}\delta_{ac}].$$

The adjoint representation transforms as follows : $45'_{ij} = O_{ik}O_{jl}45_{kl}$ or $45'_{ij} = (O45O^T)_{ij}$.

In general the tensor product $(10 \times 10) = (10 \times 10)_A + (10 \times 10)_S = 45 + 54 + 1$. The 54 dimensional representation is denoted by the symmetric tensor $54_{ij} = 54_{ji},\ Tr(54) = 0$ with transformations $54' = O54O^T$.

The spinor representation of SO(10) can be defined in terms of $2^5 \times 2^5$ dimensional representations of a Clifford algebra $\Gamma_i,\ i = 1, \cdots, 10$, just as for example the spinor representation of SO(4) is represented in terms of 4×4 Dirac gamma matrices (see for example, Georgi, "Lie Algebras in Particle Physics" for a more detailed discussion[29]). The Γs satisfy $\Gamma_i^\dagger = \Gamma_i,\ \ \{\Gamma_i,\ \Gamma_j\} = 2\delta_{ij}$. They can explicitly be expressed in terms of tensor products of 5 Pauli matrices, although we will not do this here. We can also define $\Gamma_{11} \equiv \prod_{i=1}^{10} \Gamma_i$ satisfying $\{\Gamma_{11},\ \Gamma_i\} = 0$ for all i. The generators of SO(10) in the spinor representation are now given by

$$\Sigma_{ij} = \frac{i}{4}[\Gamma_i,\ \Gamma_j].$$

Note $[\Gamma_{11},\ \Sigma_{ij}] = 0$ and $\Gamma_{11}^2 = 1$. Hence Γ_{11} has eigenvalues ± 1 which divides the 32 dimensional spinor into two irreducible representations of SO(10) which are the 16 and $\overline{16}$ spinor representations.

In order to generate some intuition on how SO(10) acts on the spinor representations, we use the gamma matrices to define operators satisfying a Heisenberg algebra of creation and annihilation operators. Let

$$A_\alpha = \frac{\Gamma_{2\alpha-1} + i\Gamma_{2\alpha}}{2},\ \ \alpha = 1, \cdots, 5$$

and

$$A_\alpha^\dagger = \frac{\Gamma_{2\alpha-1} - i\Gamma_{2\alpha}}{2}.$$

The As satisfy $\{A_\alpha,\ A_\beta\} = 0,\ \ \{A_\alpha,\ A_\beta^\dagger\} = \delta_{\alpha\beta}$. We could now rewrite the generators of SO(10) explicitly in terms of products of As and $A^\dagger s$. Instead of doing this let me directly identify an SU(5) subgroup of SO(10). In fact the set of generators $\{\Sigma_{ij}\}$ are equivalent to the set of generators $\{Q_a, \Delta_{\alpha\beta}, \Delta^\dagger_{\alpha\beta}, X\}$ defined by

$$Q_a = A_\alpha^\dagger \frac{\lambda^a_{\alpha\beta}}{2} A_\beta,\ \ a = 1, \cdots, 24$$

where $\lambda^a_{\alpha\beta}$ are the 5×5 traceless hermitian generators of SU(5) in the 5 dimensional representation. It is then easy to see that the Qs satisfy the Lie algebra of SU(5), $[Q_a,\ Q_b] = i f_{abc} Q_c$. Define

$$\Delta_{\alpha\beta} = A_\alpha A_\beta = -\Delta_{\beta\alpha},\ \ \Delta^\dagger_{\alpha\beta} = A^\dagger_\alpha A^\dagger_\beta = -\Delta^\dagger_{\beta\alpha}.$$

Finally, we define

$$X = -2 \sum_{\alpha=1}^{5} (A_\alpha^\dagger A_\alpha - \frac{1}{2}),$$

the U(1) generator which commutes with the generators of SU(5).

Let us now define the 16, $\overline{16}$ representations explicitly. Consider first the 16 which contains a $10 + \overline{5} + 1$ under SU(5). Let $|0\rangle \equiv |\mathbf{1}\rangle \equiv [0]$ be the SU(5) invariant state contained in the 16, such that $Q_a|0\rangle \equiv 0$. It is thus the vacuum state for the annihilation operators A (i.e. $A_\alpha|0\rangle \equiv 0$), an SU(5) singlet and a zero index tensor under SU(5) transformations respectively. We now have $\Delta^\dagger_{\alpha\beta}|0\rangle = |\mathbf{10}\rangle_{\alpha\beta} = [2]$ a 2 index antisymmetric tensor or 10 under SU(5). Finally, $\epsilon^{\alpha\beta\gamma\delta\lambda}\Delta^\dagger_{\alpha\beta}\Delta^\dagger_{\gamma\delta}|0\rangle = |\overline{\mathbf{5}}\rangle^\lambda = [4]$. Thus, in summary, we have defined the $16 = 10 + \overline{5} + 1$ by

$$|\mathbf{1}\rangle = |0\rangle, \;\; |\mathbf{10}\rangle_{\alpha\beta} = \Delta^\dagger_{\alpha\beta}|0\rangle, \;\; |\overline{\mathbf{5}}\rangle^\lambda = \epsilon^{\alpha\beta\gamma\delta\lambda}\Delta^\dagger_{\alpha\beta}\Delta^\dagger_{\gamma\delta}|0\rangle.$$

Similarly the $\overline{16} = \overline{10} + 5 + 1$ is defined by

$$|\mathbf{5}\rangle_\alpha = A^\dagger_\alpha|0\rangle, \;\; |\overline{\mathbf{10}}\rangle^{\delta\rho} = \epsilon^{\alpha\beta\gamma\delta\rho}\Delta^\dagger_{\alpha\beta}A^\dagger_\gamma|0\rangle, \;\; |\mathbf{1}\rangle = \epsilon^{\alpha\beta\gamma\delta\rho}\Delta^\dagger_{\alpha\beta}\Delta^\dagger_{\gamma\delta}A^\dagger_\rho|0\rangle.$$

SO(10) is a rank 5 group, meaning there are 5 U(1) generators in the Cartan subalgebra. The 5 generators can be defined as:

$$\Sigma_{12} = \frac{i}{4}[\Gamma_1, \; \Gamma_2] \equiv (A_1^\dagger A_1 - 1/2),$$

$$\Sigma_{34} = \frac{i}{4}[\Gamma_3, \; \Gamma_4] \equiv (A_2^\dagger A_2 - 1/2),$$

$$\Sigma_{56} = \frac{i}{4}[\Gamma_5, \; \Gamma_6] \equiv (A_3^\dagger A_3 - 1/2),$$

$$\Sigma_{78} = \frac{i}{4}[\Gamma_7, \; \Gamma_8] \equiv (A_4^\dagger A_4 - 1/2),$$

$$\Sigma_{9\,10} = \frac{i}{4}[\Gamma_9, \; \Gamma_{10}] \equiv (A_5^\dagger A_5 - 1/2).$$

The first 3 act on color indices and the last two act on weak indices. Thus the SU(5) invariant U(1) generator in the 16 dimensional representation is given by

$$X = -2 \sum_{\alpha=1}^{5} (A_\alpha^\dagger A_\alpha - 1/2) = -2(\Sigma_{12} + \Sigma_{34} + \Sigma_{56} + \Sigma_{78} + \Sigma_{910}).$$

The 10 dimensional representation can be expressed in terms of a $(5 \times 5) \otimes (2 \times 2)$ tensor product notation. We can use the above formula to write an expression for X in this basis. We find

$$X = 2x \otimes \eta$$

where

$$x = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\eta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Similarly we can identify the other U(1)s which commute with $SU(3)\times SU(2)\times U(1)_Y$:

$$Y = -\frac{2}{3}\sum_{\alpha=1}^{3}(A^\dagger_\alpha A_\alpha - 1/2) + \sum_{\alpha=4}^{5}(A^\dagger_\alpha A_\alpha - 1/2)|_{on16} = y \otimes \eta|_{on10}$$

where $y = \begin{pmatrix} 2/3 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$;

$$B - L = -\frac{2}{3}\sum_{\alpha=1}^{3}(A^\dagger_\alpha A_\alpha - 1/2)|_{on16} = \frac{2}{3}(b-l)\otimes\eta$$

where $(b-l) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$; and

$$T_{3R} = -\frac{1}{2}\sum_{\alpha=4}^{5}(A^\dagger_\alpha A_\alpha - 1/2)|_{on16} = \frac{1}{2}t_{3R}\otimes\eta$$

where $t_{3R} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

It is a useful exercise to use the definition of the 16 defined above and the definition of Y in terms of number operators to identify the hypercharge assignments of the states in the 16.

Note that we will use fields in the adjoint (45) representation to break SO(10) to the SM. A 45 vev in the X direction will break SO(10) to $SU(5)\times U(1)_X$. The vev of a $16+\overline{16}$ in the $\overline{\nu}$ directions can then break X leaving SU(5) invariant. We

could then use a 45 with vev in either the $Y, B-L$ or T_{3R} directions to break SU(5) to $SU(3) \times SU(2) \times U(1)_Y$. Note also that either (X, Y) or $(B-L, T_{3R})$ span the 2 dimensional space of U(1)s which commute with $SU(3) \times SU(2) \times U(1)_Y$.

Finally, the 16 of SO(10) contains one family of fermions and their supersymmetric partners. The 10 of SO(10) contains a pair of Higgs doublets necessary to do the electroweak breaking. Under SU(5) we have $10 = 5 + \overline{5}$. The simplest dimension 4 Yukawa coupling of the electroweak Higgs to a single family (consider the third generation) is given by

$$A 16_3 \; 10 \; 16_3.$$

The SO(10) symmetry relation which follows is

$$\lambda_t = \lambda_b = \lambda_\tau = \lambda_{\nu_\tau} = A.$$

6. The Effective Operator Approach

We have studied the supersymmetrized Georgi-Jarlskog texture for fermion masses. It is remarkably successful in describing the low energy data. Nevertheless there are some inherent shortcomings with the texture approach.

1. The texture of zeros is adhoc - perhaps there are others which work better or have fewer parameters.

2. Down and charged lepton Yukawa matrices are related, but *not* up quarks – this is so even for the SO(10) version of the theory.

3. There is no explanation of the family hierarchy - the arbitrary parameters simply satisfy $A >> B >> C$ and $D >> E >> F$.

4. The important Clebsch factor of 3 requires a Higgs 45 dimensional representation in SU(5) or a $\overline{126}$ in SO(10) - these are large representations which make it difficult to construct complete GUT theories.

The third problem above suggests we consider the higher dimension effective operators of Froggatt-Nielsen[7]. Combine this with a desire for maximal predictability and we are led to consider GUTs with additional family symmetries. In the remainder of these lectures we will describe an effective supersymmetric SO(10) operator analysis of fermion masses. We define a procedure for finding the dominant operator set reproducing the low energy data. In the minimal operator sets we have just six parameters in the fermion mass matrices. We use the six best known low energy parameters as input to fix these six unknowns and then predict the rest. These theories are supersymmetric[SUSY] SO(10) grand unified theories[GUTs][12]. In the next two sections I want to briefly motivate these choices.

7. Why SUSY GUTs?

Looking back at the history of particle physics, it is clear that much of our understanding comes from using symmetries. This is because, even without a complete understanding of the dynamics, symmetries can be used to relate different observables. Here too we want to correlate the known low energy data, the three gauge couplings and the fermion masses and mixing angles. We want to describe these 16 parameters in terms of fewer fundamental numbers. GUTs allow us to do just that. In fact using this symmetry we can express the low energy data as follows –

$$Observable = Input\ parameters \times Boundary\ condition\ at\ M_G \times RG\ factor$$

where the *observable* is the particular low energy data we want to calculate, the *input parameters* is the set of fundamental parameters defined at the GUT scale and the last factor takes into account the renormalization group running of the experimental observable from M_G to the low energy scale. The grand unified symmetry SU(5) (or SO(10), E(6) etc.) determines the *boundary conditions at* M_G[13]. These are given in terms of Clebsch-Gordan coefficients relating different observables. Of course, these relations are only valid at the GUT scale and the RG equations are necessary to relate them to experiment. It is through the RG equations that supersymmetry enters. We will assume that only the states present in the minimal supersymmetric standard model[MSSM] are in the theory below M_G. We assume this because it works. Consider the GUT expression for the gauge couplings –

$$\alpha_i(M_Z) = \alpha_G\ R_i(\alpha_G, \frac{M_G}{M_Z})$$

where the boundary condition is $R_i(\alpha_G, 1) \equiv 1 + \cdots$. The input parameters are α_G and M_G and the Clebschs in this case are all one. Thus we obtain the well known result that given α and $\sin^2\theta_W$ measured at M_Z we predict the value for $\alpha_s(M_Z)$[14](For recent analysis of the data, see [15]). Note that SUSY without GUTs makes no prediction, since there is no symmetry to specify the boundary conditions and GUTs without SUSY makes the wrong prediction.

I should also point out that the SO(10) operator analysis for fermion masses that I am about to describe is not new. This analysis was carried out 10 years ago with the result that the favored value of the top quark mass was about 35 GeV[31].

8. Why SO(10)?

There are two reasons for using SO(10).

1. It is the smallest group in which all the fermions in one family fit into one irreducible representation, i.e. the **16**. Only one additional state needs to be added to complete the multiplet and that is a right-handed neutrino. In larger gauge groups, more as yet unobserved states must be introduced to obtain

complete multiplets. Thus we take $\mathbf{16}_i \supset \{U_i, D_i, E_i, \nu_i\}, i = 1, 2, 3$ for the 3 families with the third family taken to be the heaviest. *Since SO(10) Clebschs can now relate U, D, E and ν mass matrices, we can in principle reduce the number of fundamental parameters in the fermion sector of the theory.* We return to this point below.

2. In any SUSY theory there are necessarily two higgs doublets – H_u and H_d. Both these states fit into the **10** of SO(10) and thus their couplings to up and down type fermions are also given by a Clebsch. There are however six additional states in the **10** which transform as a $\mathbf{3} + \overline{\mathbf{3}}$ under color. These states contribute to proton decay and must thus be heavy. The problem of giving these color triplet states large mass of order M_G while keeping the doublets light is sometimes called the second gauge hierarchy problem. This problem has a natural solution in SO(10) which we discuss later[32].

Note that the gauge group SO(10) has to be spontaneously broken to the gauge group of the SM – $SU(3) \times SU(2) \times U(1)$. This GUT scale breaking can be accomplished by a set of states including $\{\ \mathbf{45}, \mathbf{16}, \overline{\mathbf{16}}, \cdots\}$. The **45**(the adjoint representation) enters into our construction of effective fermion mass operators, thus I will discuss it in more detail in the next section.

I promised to return to the possibility of reducing the number of fundamental parameters in the fermion sector of the theory. Recall that there are 13 such parameters. Using symmetry arguments we can now express the matrices $\mathbf{D}, \mathbf{E}$, and ν in terms of one complex 3x3 matrix, U. Unfortunately, this is not sufficient to solve our problem. There are 18 arbitrary parameters in this one matrix. In order to reduce the number of fundamental parameters we must have zeros in this matrix. We thus need new *family symmetries* to enforce these zeros.

9. The Big Picture

Let us consider the big picture(see Fig. 10). Our low energy observer measures the physics at the electroweak scale and perhaps an order of magnitude above. Once the SUSY threshold is crossed we have direct access to the effective theory at M_G, the scale where the 3 gauge couplings meet. Of course the GUT scale $M_G \sim 10^{16}$ GeV is still one or two orders of magnitude below some more fundamental scale such as the Planck or string scales (which we shall refer to as M). Between M and M_G there may be some substructure. In fact, we may be able to infer this substructure by studying fermion masses.

In our analysis we assume that the theory below the scale M is described by a SUSY SO(10) GUT. Between M_G and M, at a scale v_{10}, we assume that the gauge group SO(10) is broken spontaneously to SU(5). This can occur due to the vacuum expectation value of an adjoint scalar in the X direction and the expectation values of a 16 and a $\overline{16}$(denoted by Ψ and $\overline{\Psi}$ respectively). Then SU(5) is broken at the scale $v_5 = M_G$ to the SM gauge group. This latter breaking can be done by different adjoints (45) in the Y, B-L or T_{3R} directions.

Why consider 4 particular breaking directions for the 45 and no others? The X and Y directions are orthogonal and span the two dimensional space of U(1) subgroups of SO(10) which commute with the SM. B-L and T_{3R} are also orthogonal and they span the same subspace. Nevertheless we consider these four possible breaking directions and these are the *only directions* which will enter the effective operators for fermion masses. Why not allow the X and Y directions or any continous rotation of them in this 2d subspace of U(1) directions . The answer is that there are good dynamical arguments for assuming that these and only these directions are important. The X direction breaks SO(10) to an intermediate SU(5) subgroup and it is reasonable to assume that this occurs at a scale $v_{10} \geq v_5$. Whether v_{10} is greater than v_5 or equal will be determined by the LED. The B-L direction is required for other reasons. Recall the color triplet higgs in the 10 which must necessarily receive large mass. As shown by Dimopoulos and Wilczek[32], this doublet-triplet splitting can naturally occur by introducing a 10 45 10 type coupling in the superspace potential. Note that the higgs triplets carry non-vanishing B-L charge while the doublets carry zero charge. Thus when the 45 gets a vacuum expectation value[vev] in the B-L direction it will give mass to the color triplet higgs at v_5 and leave the doublets massless. Thus in any SO(10) model which solves this second hierarchy problem, there must be a 45 pointing in the B-L direction. We thus allow for all 4 possible breaking vevs — X, Y, B-L and T_{3R}. Furthermore we believe this choice is "natural" since we know how to construct theories which have these directions as vacua without having to tune any parameters.

Our fermion mass operators have dimension ≥ 4. *From where would these higher dimension operators come?* Note that by measuring the LED we directly probe the physics in some effective theory at M_G. This effective theory can, and likely will, include operators with dimension greater than 4. Consider, for example, our big picture looking down from above. String theories are very fundamental. They can in principle describe physics at all scales. Given a particular string vacuum, one can obtain an effective field theory valid below the string scale M. The massless sector can include the gauge bosons of SO(10) with scalars in the 10, 45 or even 54 dimensional representations. In addition, we require 3 families of fermions in the 16. Of course, in a string context when one says that there are 3 families of fermions what is typically meant is that there are 3 more 16s than $\overline{16}$s. The extra 16 + $\overline{16}$ pairs are assumed to get mass at a scale $\geq M_G$, since there is no symmetry which prevents this. When these states are integrated out in order to define the effective field theory valid below M_G they will typically generate higher dimension operators.

Consider a simple two family model. Let $16_2,\ 16_3$ represent the 2 heaviest families of quarks and leptons, $\Psi_i,\ \overline{\Psi}_i,\ \ i = 1, 2$ are heavy $16, \overline{16}$ states with mass of order M_G, $A_2, \tilde{A}$ are in the 45 dimensional representation and 10 contains the electroweak Higgs doublets. In this example we have 4 $16s$ and 2 $\overline{16}s$. At the scale M we assume the superspace potential has the form

$$16_3 10 16_3 + g_3 16_3 A_2 \overline{\Psi}_1 + \tilde{g}_1 \overline{\Psi}_1 \tilde{A} \Psi_1 + g_2 16_2 A_2 \overline{\Psi}_2 + \tilde{g}_2 \overline{\Psi}_2 \tilde{A} \Psi_2 + \Psi_1 10 \Psi_2.$$

We now assume that $\langle \tilde{A} \rangle \sim X$ and $\langle A_2 \rangle \sim Y$ with $\langle \tilde{A} \rangle >> \langle A_2 \rangle$. Thus the dominant

contribution to the mass of the states $\Psi_i,\ \overline{\Psi}_i,\ \ i=1,2$ is given by $\tilde{g}_i\langle\tilde{A}\rangle$. In order to define the effective theory at M_G, we must integrate these states out of the theory. As a result we obtain the effective mass operators -

$$O_{33} = 16_3^p 10^{-2p} 16_3^p, \quad O_{32} = 16_3^p \left(\frac{A_2}{\tilde{A}}\right)^{q_2-t} 10^{-2p} \left(\frac{A_2}{\tilde{A}}\right)^{q_2-t} 16_2^{p-2q_2+2t}$$

which can be read off the tree diagrams in fig. 11. The superscripts in this formula denote independent U(1) charges which may be assigned to the fields. The sum of the charges at any vertex must vanish for $U(1)_p, U(1)_{q_2}, U(1)_t$ to be symmetries of the theory. Note, at the level of the effective operators, the operator

$$16_3^p \left(\frac{A_2}{\tilde{A}}\right)^{2q_2-2t} 10^{-2p} 16_2^{p-2q_2+2t}$$

also preserves all 3 U(1) symmetries. This operator is not equivalent to O_{32} above. It cannot be obtained however by integrating out the heavy fields. Thus the symmetries of the full theory restrict the order of operators appearing in the effective theory.

The operators O_{33} and O_{32} represent only the first term in a power series in the ratio $\left|\frac{\langle A_2\rangle}{\langle\tilde{A}\rangle}\right|^2$. We can obtain the complete effective theory by diagonalizing the 4×2 mass matrix

$$\begin{array}{c c} & \begin{array}{cccc} \Psi_1 & \Psi_2 & 16_3 & 16_2 \end{array} \\ \begin{array}{c} \overline{\Psi}_1 \\ \overline{\Psi}_2 \end{array} & \left(\begin{array}{cccc} \tilde{g}_1\langle\tilde{A}\rangle & 0 & g_3\langle A_2\rangle & 0 \\ 0 & \tilde{g}_2\langle\tilde{A}\rangle & 0 & g_2\langle A_2\rangle \end{array}\right) \end{array}.$$

The mass eigenstates are given by

$$\Psi_1' = (\tilde{g}_1\langle\tilde{A}\rangle\Psi_1 + g_3\langle A_2\rangle 16_3)/m_1,$$

$$16_3' = (-g_3\langle A_2\rangle\Psi_1 + \tilde{g}_1\langle\tilde{A}\rangle 16_3)/m_1$$

where $m_1 = \sqrt{\tilde{g}_1^2|\langle\tilde{A}\rangle|^2 + g_3^2|\langle A_2\rangle|^2}$. Similarly,

$$\Psi_2' = (\tilde{g}_2\langle\tilde{A}\rangle\Psi_2 + g_2\langle A_2\rangle 16_2)/m_2,$$

$$16_2' = (-g_2\langle A_2\rangle\Psi_2 + \tilde{g}_2\langle\tilde{A}\rangle 16_2)/m_2$$

where $m_2 = \sqrt{\tilde{g}_2^2|\langle\tilde{A}\rangle|^2 + g_2^2|\langle A_2\rangle|^2}$. The states $16_3', 16_2'$ are massless, while the other states have mass terms $\sum_{i=1}^2(m_i\overline{\Psi}_i\Psi_i')$. We can now invert the relations to get

$$16_3 = (\tilde{g}_1\langle\tilde{A}\rangle 16_3' + g_3\langle A_2\rangle\Psi_1')/m_1,$$

$$\Psi_1 = (\tilde{g}_1\langle\tilde{A}\rangle\Psi_1' - g_3\langle A_2\rangle 16_3')/m_1,$$

$$\Psi_2 = (\tilde{g}_2\langle\tilde{A}\rangle\Psi_2' - g_2\langle A_2\rangle 16_2')/m_2.$$

The effective field theory is now obtained by taking the terms in the superspace potential $16_3 10 16_3 + \Psi_1 10 \Psi_2$ and replacing $16_3, \Psi_1, \Psi_2$ by their massless components. We find

$$16_3 \left(\frac{1}{\sqrt{1 + \left| \frac{g_3 \langle A_2 \rangle}{\tilde{g}_1 \langle \tilde{A} \rangle} \right|^2}} \right) 10 \left(\frac{1}{\sqrt{1 + \left| \frac{g_3 \langle A_2 \rangle}{\tilde{g}_1 \langle \tilde{A} \rangle} \right|^2}} \right) 16_3$$

$$+16_3 \left(\frac{g_3 \langle A_2 \rangle}{\tilde{g}_1 \langle \tilde{A} \rangle} \right) \left(\frac{1}{\sqrt{1 + \left| \frac{g_3 \langle A_2 \rangle}{\tilde{g}_1 \langle \tilde{A} \rangle} \right|^2}} \right) 10 \left(\frac{g_2 \langle A_2 \rangle}{\tilde{g}_2 \langle \tilde{A} \rangle} \right) \left(\frac{1}{\sqrt{1 + \left| \frac{g_2 \langle A_2 \rangle}{\tilde{g}_2 \langle \tilde{A} \rangle} \right|^2}} \right) 16_2 .$$

10. Operator Basis for Fermion Masses at M_G

Let us now consider the general **operator basis for fermion masses**. We include operators of the form

$$\mathbf{O_{ij}} = \mathbf{16_i} \; (\cdots)_n \; \mathbf{10} \; (\cdots)_m \; \mathbf{16_j}$$

where

$$(\cdots)_n = \frac{M_G^k \; 45_{k+1} \cdots 45_n}{M_P^l \; 45_X^{n-l}}$$

and the 45 vevs in the numerator can be in any of the 4 directions, $\mathbf{X}, \mathbf{Y}, \mathbf{B-L}, \mathbf{T_{3R}}$ discussed earlier.

It is trivial to evaluate the Clebsch-Gordon coefficients associated with any particular operator since the matrices $X, Y, B-L, T_{3R}$ are diagonal. Their eigenvalues on the fermion states are given in Table 2.

It is probably useful to consider a couple of examples of effective operators and work out their contributions to fermion mass matrices before we continue with our discussion of the systematic search over all operator sets which are consistent with the low energy data. For our first example consider the 2 family effective theory discussed earlier. The superspace potential is given by

$$O_{33} + O_{32} = 16_3 10 16_3 + 16_3 \left(\frac{\langle A_2 \rangle}{\langle \tilde{A} \rangle} \right) 10 \left(\frac{\langle A_2 \rangle}{\langle \tilde{A} \rangle} \right) 16_2 .$$

We now assume $\langle A_2 \rangle = a_2 e^{i\alpha_2} Y, \;\; \langle \tilde{A} \rangle = \tilde{a} e^{i\tilde{\alpha}} X$ with $a_2 \sim M_G$ and $\tilde{a} = v_{10} > M_G$.

We can now evaluate the Yukawa matrices. We find

$$\mathcal{U} = \begin{pmatrix} 0 & x'_u B \\ x_u B & A \end{pmatrix},$$

$$\mathcal{D} = \begin{pmatrix} 0 & x'_d B \\ x_d B & A \end{pmatrix},$$

Table 2. Quantum numbers of the four 45 vevs on fermion states. Note, if u denotes a left-handed up quark, then $\overline{u}$ denotes a left-handed charge conjugate up quark.

	$\mathbf{X}$	$\mathbf{Y}$	$\mathbf{B-L}$	$\mathbf{T_{3R}}$
u	1	1/3	1	0
$\overline{u}$	1	-4/3	-1	-1/2
d	1	1/3	1	0
$\overline{d}$	-3	2/3	-1	1/2
e	-3	-1	-3	0
$\overline{e}$	1	2	3	1/2
ν	-3	-1	-3	0
$\overline{\nu}$	5	0	3	-1/2

$$\mathcal{E} = \begin{pmatrix} 0 & x_e' B \\ x_e B & A \end{pmatrix}.$$

The constant B is given in terms of ratio of scales

$$B = (\text{ratio of coupling constants}) \left(\frac{a_2}{\tilde{a}}\right)^2$$

where we have explicitly redefined the phases of fermions to remove the arbitrary phase. Finally we evaluate the Clebschs

$$x_u = x_u' = -4/9, x_d = x_d' = -2/27, x_e = x_e' = 2/3.$$

A particular ratio of Clebschs

$$\chi \equiv \frac{|x_u - x_d|}{\sqrt{|x_u x_u'|}} = 5/6.$$

In this case the Yukawa matrices are symmetric.

In the next example, replace the operator O_{32} above by

$$O_{32} = 16_3 \frac{A_1}{\tilde{A}} 10 \frac{A_2}{\tilde{A}} 16_2$$

where $\langle A_1 \rangle = a_1 e^{i\alpha_1}(B-L)$. In this case $B \approx \left(\frac{a_1 a_2}{\tilde{a}^2}\right)$. We find the Clebschs

$$x_u = -1/3, x_u' = -4/3, x_d = 1/9, x_d' = -2/9, x_e = 1, x_e' = 2$$

and in this case

$$\chi \equiv \frac{|x_u - x_d|}{\sqrt{|x_u x_u'|}} = 2/3.$$

You see that the Yukawa matrices are no longer symmetric.

11. Dynamic Principles

Now consider the dynamical principles which guide us towards a theory of fermion masses.

0. At zeroth order, we work in the context of a SUSY GUT with the MSSM below M_G.

1. We use SO(10) as the GUT symmetry with three families of fermions $\{16_i \;\; i = 1,2,3\}$ and the minimal electroweak Higgs content in one 10. SO(10) symmetry relations allow us to reduce the number of fundamental parameters.

2. We assume that there are also family symmetries which enforce zeros of the mass matrix, although we will not specify these symmetries at this time. As we will make clear in section 12, these symmetries will be realized at the level of the fundamental theory defined below M.

3. Only the third generation obtains mass via a dimension 4 operator. The fermionic sector of the Lagrangian thus contains the term $A\ O_{33} \equiv A\ \ 16_3\ 10\ 16_3$. This term gives mass to t, b and τ. It results in the symmetry relation — $\lambda_t = \lambda_b = \lambda_\tau \equiv A$ at M_G. This relation has been studied before by Ananthanarayan, Lazarides and Shafi[28] and using m_b and m_τ as input it leads to reasonable results for m_t and $\tan\beta$.

4. All other masses come from operators with dimension > 4. As a consequence, the family hierarchy is related to the ratio of scales above M_G.

5. [**Predictivity requirement**] We demand the <u>minimal set</u> of effective fermion mass operators at M_G <u>consistent with the **LED**</u>.

12. Systematic Search

Our goal is to find the *minimal* set of fermion mass operators consistent with the LED. With any given operator set one can evaluate the fermion mass matrices for up and down quarks and charged leptons. One obtains relations between mixing angles and ratios of fermion masses which can be compared with the data. It is easy to show, however, without any detailed calculations that the minimal operator set consistent

with the LED is given by

$$O_{33} + O_{23} + O_{22} + O_{12} \quad - - - \text{“22” texture}$$

or

$$O_{33} + O_{23} + O'_{23} + O_{12} \quad - - - \text{“23′” texture}$$

It is clear that at least 3 operators are needed to give non-vanishing and unequal masses to all charged fermions, i.e. $det(m_a) \neq 0$ for $a = u, d, e$. That the operators must be in the [33, 23 and 12] slots is not as obvious but is not difficult to show. It is then easy to show that 4 operators are required in order to have CP violation. This is because, with only 3 SO(10) invariant operators, we can redefine the phases of the three 16s of fermions to remove the three arbitrary phases. With one more operator, there is one additional phase which cannot be removed. A corollary of this observation is that this minimal operator set results in just 5 arbitrary parameters in the Yukawa matrices of all fermions, 4 magnitudes and one phase.‡ This is the minimal parameter set which can be obtained without solving the remaining problems of the fermion mass hierarchy, one overall real mixing angle and a CP violating phase. We should point out however that the problem of understanding the fermion mass hierarchy and mixing has been rephrased as the problem of understanding the hierarchy of scales above M_G.

From now on I will just consider models with “22” texture. This is because they can reproduce the observed hierarchy of fermion masses without fine-tuning.§ Models with “22” texture give the following Yukawa matrices at M_G (with electroweak doublet fields on the right) –

$$\lambda_{\mathbf{a}} = \begin{pmatrix} 0 & z'_a\ C & 0 \\ z_a\ C & y_a\ E\ e^{i\phi} & x'_a\ B \\ 0 & x_a\ B & A \end{pmatrix}$$

with the subscript $a = \{u, d, e\}$. The constants $x_a, x'_a, y_a, z_a, z'_a$ are Clebschs which can be determined once the 3 operators (O_{23}, O_{22}, O_{12}) are specified. Recall, we have taken $O_{33} = A\ 16_3\ 10\ 16_3$, which is why the Clebsch in the 33 term is independent of a. Finally, combining the Yukawa matrices with the Higgs vevs to find the fermion mass matrices we have 6 arbitrary parameters given by A, B, C, E, ϕ and $\tan\beta$ describing 14 observables. We thus obtain 8 predictions. We shall use the best known parameters, $m_e, m_\mu, m_\tau, m_c, m_b, |V_{cd}|$, as input to fix the 6 unknowns. We then predict the values of $m_u, m_d, m_s, m_t, \tan\beta, |V_{cb}|, |V_{ub}|$ and J.

Note: since the predictions are correlated, our analysis would be much improved if we minimized some χ^2 distribution and obtained a best fit to the data. Unfortunately this has not yet been done. In the paper however we do include some tables (see for example Table 5) which give all the predictions for a particular set of input parameters.

‡This is two fewer parameters than was necessary in our previous analysis (see [21])

§For more details on this point, see section 14 below or refer to [30].

13. Results

The results for the 3rd generation are given in fig. 12. Note that since the parameter A is much bigger than the others we can essentially treat the 3rd generation independently. The small corrections, of order $(B/A)^2$, are however included in the complete analysis. We find the pole mass for the top quark $M_t = 180 \pm 15$ GeV and $\tan\beta = 56 \pm 6$ where the uncertainties result from variations of our input values of the $\overline{MS}$ running mass $m_b(m_b) = 4.25 \pm 0.15$ and $\alpha_s(M_Z)$ taking values $.110 - .126$. We used two loop RG equations for the MSSM from M_G to M_{SUSY}; introduced a universal SUSY threshold at $M_{SUSY} = 180$ GeV with 3 loop QCD and 2 loop QED RG equations below M_{SUSY}. The variation in the value of α_s was included to indicate the sensitivity of our results to threshold corrections which are necessarily present at the weak and GUT scales. In particular, we chose to vary $\alpha_s(M_Z)$ by letting $\alpha_3(M_G)$ take on slightly different values than $\alpha_1(M_G) = \alpha_2(M_G) = \alpha_G$.

The following set of operators passed a straightforward but coarse grained search discussed in detail in the paper[30]. They include the diagonal dimension four coupling of the third generation –

$$O_{33} = \; 16_3 \; 10 \; 16_3$$

.

The six possible O_{22} operators –

$$\begin{array}{lcr} O_{22} = & & \\ & 16_2 \, \frac{45_X}{M} \, 10 \, \frac{45_{B-L}}{45_X} \, 16_2 & (a) \\ & 16_2 \, \frac{M_G}{45_X} \, 10 \, \frac{45_{B-L}}{M} \, 16_2 & (b) \\ & 16_2 \, \frac{45_X}{M} \, 10 \, \frac{45_{B-L}}{M} \, 16_2 & (c) \\ & 16_2 \, 10 \, \frac{45_{B-L}}{45_X} \, 16_2 & (d) \\ & 16_2 \, 10 \, \frac{45_X \, 45_{B-L}}{M^2} \, 16_2 & (e) \\ & 16_2 \, 10 \, \frac{45_{B-L} \, M_G}{45_X^2} \, 16_2 & (f) \end{array}$$

Note: in all cases the Clebschs y_i (defined by O_{22} above) satisfy

$$y_u : y_d : y_e = 0 : 1 : 3.$$

This is the form familiar from the Georgi-Jarlskog texture[19]. Thus all six of these operators lead to *identical* low energy predictions.

Finally there is a unique operator O_{12} consistent with the LED –

$$O_{12} = \; 16_1 \left(\frac{45_X}{M}\right)^3 \, 10 \left(\frac{45_X}{M}\right)^3 16_2$$

The operator O_{23} determines the KM element V_{cb} by the relation –

$$V_{cb} = \chi \sqrt{\frac{m_c}{m_t}} \times (RG factors)$$

where the Clebsch combination χ is given by

$$\chi \equiv \frac{|x_u - x_d|}{\sqrt{|x_u x'_u|}}$$

m_c is input, m_t has already been determined and the RG factors are calculable. Demanding the experimental constraint $V_{cb} < .054$ we find the constraint $\chi < 1$. A search of all operators of dimension 5 and 6 results in the 9 operators given below. Note that there only three different values of $\chi = 2/3,\ 5/6,\ 8/9$ –

$$O_{23} =$$

$$\chi = 2/3$$

$$(1) \quad 16_2 \, \frac{45_Y}{M} \, 10 \frac{M_G}{45_X} \, 16_3$$

$$(2) \quad 16_2 \, \frac{45_Y}{M} \, 10 \, \frac{45_{B-L}}{45_X} \, 16_3$$

$$(3) \quad 16_2 \, \frac{45_Y}{45_X} \, 10 \, \frac{M_G}{45_X} \, 16_3$$

$$(4) \quad 16_2 \, \frac{45_Y}{45_X} \, 10 \, \frac{45_{B-L}}{45_X} 16_3$$

$$\chi = 5/6$$

$$(5) \quad 16_2 \, \frac{45_Y}{M} \, 10 \, \frac{45_Y}{45_X} \, 16_3$$

$$(6) \quad 16_2 \, \frac{45_Y}{45_X} \, 10 \, \frac{45_Y}{45_X} \, 16_3$$

$$\chi = 8/9$$

$$(7) \quad 16_2 \, 10 \, \frac{M_G^2}{45_X^2} \, 16_3$$

$$(8) \quad 16_2 \, 10 \, \frac{45_{B-L} M_G}{45_X^2} \, 16_3$$

$$(9) \quad 16_2 \, 10 \, \frac{45_{B-L}^2}{45_X^2} \, 16_3$$

We label the operators (1) - (9), and we use these numbers also to denote the corresponding models. *Note, all the operators have the vev* 45_X *in the denominator. This can only occur if* $v_{10} > M_G$.

At this point, there are no more simple criteria to reduce the number of models further. We have thus performed a numerical RG analysis on each of the 9 models (represented by the 9 distinct operators O_{23} with their calculable Clebschs $x_a, x'_a, a = u, d, e$ along with the unique set of Clebschs determined by the operators O_{33}, O_{22} and

O_{12}). We then iteratively fit the 6 arbitrary parameters to the six low energy inputs and evaluate the predictions for each model as a function of the input parameters. The results of this analysis are given in figs. (13 - 19).

Let me make a few comments. Light quark masses (u,d,s) are $\overline{MS}$ masses evaluated at 1 GeV while heavy quark masses (c,b) are evaluated at (m_c, m_b) respectively. Finally, the top quark mass in fig. 12 is the pole mass. figs. 13 and 14 are self evident. In fig. 15, we show the correlations for two of our predictions. The ellipse in the m_s/m_d vs. m_u/m_d plane is the allowed region from chiral Lagrangian analysis[33]. One sees that we favor lower values of $\alpha_s(M_Z)$. For each fixed value of $\alpha_s(M_Z)$, there are 5 vertical line segments in the V_{cb} vs. m_u/m_d plane. Each vertical line segment represents a range of values for m_c (with m_c increasing moving up) and the different line segments represent different values of m_b (with m_b increasing moving to the left). In Figure 18 we test our agreement with the observed CP violation in the K system. The experimentally determined value of $\epsilon_K = 2.26 \times 10^{-3}$. Theoretically it is given by an expression of the form $B_K \times \{m_t, V_{ts}, \cdots\}$. B_K is the so-called Bag constant which has been determined by lattice calculations to be in the range $B_K = .7 \pm .2$[34]. In fig. 18 we have used our predictions for fermion masses and mixing angles as input, along with the experimental value for ϵ_K, and fixed B_K for the 9 different models. One sees that model 4 is inconsistent with the lattice data. In fig. 19 we present the predictions for each model, for the CP violating angles which can be measured in B decays. The interior of the "whale" is the range of parameters consistent with the SM found by Nir and Sarid[35] and the error bars represent the accuracy expected from a B factory.

Note that model 4 appears to give too little CP violation and model 9 has uncomfortably large values of V_{cb}. Thus these models are presently disfavored by the data. I will thus focus on model 6 in the rest of these lectures.

14. Summary

We have performed a systematic operator analysis of fermion masses in an effective SUSY SO(10) GUT. We use the LED to lead us to the theory. Presently there are 3 models (models 4, 6 & 9) with "22" texture which agree best with the LED, although as mentioned above model 6 is favored. In all cases we used the values of α and $\sin^2\theta_W$ (modulo threshold corrections) to fix $\alpha_s(M_Z)$.

Table 3 shows the virtue of the "22" texture. In the first column are the four operators. In the 2nd and 3rd columns are the parameters in the mass matrix relevant for that particular operator and the input parameters which are used to fix these parameters. Finally the 4th column contains the predictions obtained at each level. One sees that each family is most sensitive to a different operator.¶

Consider the theoretical uncertainties inherent in our analysis.

1. The experimentally determined values of m_b, m_c, and $\alpha_s(M_Z)$ are all subject to strong interaction uncertainties of QCD. In addition, the predicted value

¶This property is not true of "23" textures.

Table 3. Virtue of "22" texture.

Operator	Parameters	Input	Predictions
O_{33}	$\tan\beta$ A	b τ	t $\tan\beta$
O_{23}	B	c	V_{cb}
O_{22}	E	μ	s
O_{12}	C ϕ	e V_{us}	u d $\frac{V_{ub}}{V_{cb}}$ J

of $\alpha_s(M_Z)$ from GUTs is subject to threshold corrections at M_W which can only be calculated once the SUSY spectrum is known and at M_G which requires knowledge of the theory above M_G. We have included these uncertainties (albeit crudely) explicitly in our analysis.

2. In the large $\tan\beta$ regime in which we work there may be large SUSY loop corrections which will affect our results. The finite corrections to the b and τ Yukawa couplings have been evaluated[36,37]. They depend on ratios of soft SUSY breaking parameters and are significant in certain regions of parameter space.[‖] In particular it has been shown that the top quark mass can be reduced by as much as 30%. Note that although the prediction of fig. 3 may no longer be valid, there is still necessarily a prediction for the top quark mass. It is now however sensitive to the details of the sparticle spectrum and to the process of radiative electroweak symmetry breaking[38]. This means that the observed top quark mass can now be used to set limits on the sparticle spectrum. This analysis has not been done. Moreover, there are also similar corrections to the Yukawa couplings for the s and d quarks and for e and μ. These corrections are expected to affect the predictions for V_{cb}, m_s, m_u, m_d. It will be interesting to see the results of this analysis.

3. The top, bottom and τ Yukawa couplings can receive threshold corrections at M_G. We have not studied the sensitivity to these corrections.

4. Other operators could in principle be added to our effective theory at M_G. They might have a dynamical origin. We have assumed that, if there, they are subdominant. Two different origins for these operators can be imagined. The first is field theoretic. The operators we use would only be the leading terms in a power series expansion when defining an effective theory at M_G by integrating out heavier states. The corrections to these operators are expected to be about 10%. We may also be sensitive to what has commonly been referred to as Planck slop[39], operators suppressed by some power of the Planck (or string) scale M. In fact the operator O_{12} may be thought of as such. The question is why aren't

[‖]There is a small range of parameter space in which our results are unchanged[36]. This requires threshold corrections at M_G which distinguish the two Higgs scalars.

our results for the first and perhaps the second generation, hopelessly sensitive to this unknown physics? This question will be addressed in the next section.

15. Where are we going?

In the first half of Table 4 I give a brief summary of the good and bad features of the effective SUSY GUT discussed earlier. Several models were found with just four operators at M_G which successfully fit the low energy data. If we add up all the necessary parameters needed in these models we find just 12. This should be compared to the SM with 18 or the MSSM with 21. Thus these theories, minimal effective SUSY GUTs[MESG], are doing quite well. Of course the bad features of the MESG is that it is not a fundamental theory. In particular there are no symmetries which prevent additional higher dimension operators to spoil our results. Neither are we able to calculate threshold corrections, even in principle, at M_G.

It is for these reasons that we need to be able to take the MESG which best describes the LED and use it to define an effective field theory valid at scales $\leq M$. The good and bad features of the resulting theory are listed in the second half of Table 4.

- In the effective field theory below M we must incorporate the *symmetries* which guarantee that we reproduce the MESG with no additional operators[**]

  Moreover, the necessary combination of discrete, U(1) or R symmetries may be powerful enough to restrict the appearance of Planck slop.

- Finally, the *GUT symmetry breaking* sector must resolve the problems of natural doublet-triplet splitting (the second hierarchy problem), the μ problem, and give predictions for proton decay, neutrino masses and calculable threshold corrections at M_G.

- On the bad side, it is still not a fundamental theory and there may not be a unique extension of the MESG to higher energies.

16. String Threshold at M_S

Upon constructing the effective field theory $\leq M_S$, we will have determined the necessary SO(10) states, symmetries and couplings which reproduce our fermion mass relations. This theory can be the starting point for constructing a realistic string model. String model builders could try to obtain a string vacuum with a massless spectra which agrees with ours. Of course, once the states are found the string will determine the symmetries and couplings of the theory. It is hoped that in this way a

[**]This statement excludes the unavoidable higher order field theoretic corrections to the MESG which are, in principle, calculable.

Table 4.

	Good	Bad
Eff. F.T. $\leq M_G$	<u>4 op's. at $M_G \Rightarrow$ **LED**</u> 5 para's. $\Rightarrow$ 13 observables + 2 gauge para's. $\Rightarrow$ 3 observables <u>+ 5</u> soft SUSY breaking para's. $\Rightarrow \cdots$ Total **12** parameters	<u>Not fundamental</u> <u>No symmetry</u> $\Rightarrow$ Why these operators? (F.T. + Planck slop) <u>Threshold corrections?</u>
Eff. F.T. $\leq M$ $M = M_{string}$ or M_{Planck}	<u>Symmetry</u> i) gives Eff. F.T. $\leq M_G$ + corrections ii) constrains other operators <u>GUT symmetry breaking</u> i) d - t splitting ii) μ problem iii) proton decay iv) neutrino masses v) threshold corrections at M_G	<u>Not fundamental</u> <u>Not unique?</u>

fundamental theory of Nature can be found. Work in this direction by several groups is in progress[40]. String theories with SO(10), three families plus additional 16 + $\overline{16}$ pairs, 45's, 10's and even some 54 dimensional representations appear possible. One of the first results from this approach is the fact that only one of the three families has diagonal couplings to the 10, just as we have assumed.

17. Constructing the Effective Field Theory below M_S

In this section I will discuss some preliminary results obtained in collaboration with Lawrence Hall[41]. I will describe the necessary ingredients for constructing model 6. Some very general results from this exercise are already apparent.

- *States* — We have constructed a SUSY GUT which includes all the states necessary for GUT symmetry breaking and also for generating the 45 vevs in the desired directions. A minimal representation content below M_S includes 54s + 45s + 3 16s + n($\overline{16}$ + 16) pairs + 2 10s.

- *Symmetry* — In order to retain sufficient symmetry the superspace potential in the visible sector W necessarily has a number of flat directions. In particular the scales v_5 and v_{10} can only be determined when soft SUSY breaking and quantum corrections are included. An auxiliary consequence is that the vev of $W_{visible}$ vanishes in the supersymmetric limit.

- *Couplings* — As an example of the new physics which results from this analysis I will show how a solution to the μ problem, the ratio λ_b/λ_t and proton decay may be inter-related.

In Table 5 are presented the predictions for Model 6 for particular values of the input parameters.

Table 5: Particular Predictions for Model 6 with $\alpha_s(M_Z) = 0.115$

Input Quantity	Input Value	Predicted Quantity	Predicted Value
$m_b(m_b)$	4.35 GeV	M_t	176 GeV
$m_\tau(m_\tau)$	1.777 GeV	$\tan\beta$	55
$m_c(m_c)$	1.22 GeV	V_{cb}	.048
m_μ	105.6 MeV	V_{ub}/V_{cb}	.059
m_e	0.511 MeV	$m_s(1GeV)$	172 MeV
V_{us}	0.221	$\hat{B}_K$	0.64
		m_u/m_d	0.64
		m_s/m_d	24.

In addition to these predictions, the set of inputs in Table 5 predicts:
$\sin 2\alpha = -.46$, $\sin 2\beta = .49$, $\sin 2\gamma = .84$, and $J = 2.6 \times 10^{-5}$.

Model 6

The superspace potential for Model 6 has several pieces - $W = W_{fermion} + W_{symmetry\ breaking} + W_{Higgs} + W_{neutrino}$.

17.1. Fermion sector

The first term must reproduce the four fermion mass operators of Model 6. They are given by

$$\begin{aligned}
O_{33} &= \quad 16_3\ 10_1\ 16_3 \\
O_{23} &= \quad 16_2\ \tfrac{A_2}{\tilde{A}}\ 10_1\ \tfrac{A_2}{\tilde{A}}\ 16_3 \\
O_{22} &= \quad 16_2\ \tfrac{\tilde{A}}{M}\ 10_1\ \tfrac{A_1}{\tilde{A}}\ 16_2 \\
O_{12} &= \quad 16_1 \left(\tfrac{\tilde{A}}{M}\right)^3\ 10_1 \left(\tfrac{\tilde{A}}{M}\right)^3 16_2
\end{aligned}$$

There are two 10s in this model, denoted by $10_i, i = 1, 2$ but only 10_1 couples to the ordinary fermions. The A fields are different 45s which are assumed to have vevs in the following directions – $\langle A_2 \rangle = 45_Y$, $\langle A_1 \rangle = 45_{B-L}$, and $\langle \tilde{A} \rangle = 45_X$. As noted earlier, there are 6 choices for the 22 operator and we have just chosen one of them, labelled a, arbitrarily here. In figure 20, we give the tree diagrams which reproduce the effective operators for Model 6 to leading order in an expansion in the

ratio of small to large scales. The states $\overline{\Psi}_a, \Psi_a, a = 1, \cdots, 9$ are massive $\overline{16}, 16$ states respectively with mass given by $\langle \mathcal{S}_M \rangle \sim M$. Each vertex represents a separate Yukawa interaction in W$_{fermion}$ (see below). Field theoretic corrections to the effective GUT operators may be obtained by diagonalizing the mass matrices for the heavy states and integrating them out of the theory.

$$W_{fermion} =$$

$$16_3 16_3 10_1 + \overline{\Psi}_1 A_2 16_3 + \overline{\Psi}_1 \tilde{A} \Psi_1 + \Psi_1 \Psi_2 10_1$$

$$+ \overline{\Psi}_2 \tilde{A} \Psi_2 + \overline{\Psi}_2 A_2 16_2 + \overline{\Psi}_3 A_1 16_2$$

$$+ \overline{\Psi}_3 \tilde{A} \Psi_3 + \Psi_3 \Psi_4 10_1 + \mathcal{S}_M \sum_{a=4}^{9} (\overline{\Psi}_a \Psi_a)$$

$$+ \overline{\Psi}_4 \tilde{A} 16_2 + \overline{\Psi}_5 \tilde{A} \Psi_4 + \overline{\Psi}_6 \tilde{A} \Psi_5$$

$$+ \Psi_6 \Psi_7 10_1 + \overline{\Psi}_7 \tilde{A} \Psi_8 + \overline{\Psi}_8 \tilde{A} \Psi_9 + \overline{\Psi}_9 \tilde{A} 16_1$$

- *Note that the vacuum insertions in the effective operators above cannot be rearranged, otherwise an inequivalent low energy theory would result. In order to preserve this order naturally we demand that each field carries a different value of a U(1) family charge (see fig. 20).* Note also that the particular choice of a 22 operator will affect the allowed U(1) charges of the states. Some choices may be acceptable and others not.
- Consider W$_{fermion}$. It has many terms, each of which can have different, in principle, complex Yukawa couplings. *Nevertheless the theory is predictive because only a very special linear combination of these parameters enters into the effective theory at M_G. Thus the observable low energy world is simple, not because the full theory is particularly simple, but because the symmetries are such that the effective low energy theory contains only a few dominant terms.*

17.2. Symmetry breaking sector

The symmetry breaking sector of the theory is not particularly illuminating. Two 54 dimensional representations, S, S' are needed plus several singlets denoted by $\mathcal{S}_i, i = 1, \cdots, 7$. They appear in the first two terms and are responsible for driving the vev of A_1 into the B-L direction, the third term drives the vev of the $\overline{16}, 16$ fields $\overline{\Psi}, \Psi$ into the right-handed neutrino like direction breaking SO(10) to SU(5)

and forcing $\tilde{A}$ into the X direction. The fourth, fifth and sixth terms drive A_2 into the Y direction. Finally the last two terms are necessary in order to assure that all non singlet states under the SM gauge interactions obtain mass of order the GUT scale. All primed fields are assumed to have vanishing vevs.

Note if $\langle \mathcal{S}_3 \rangle \approx M_S$ then two of these adjoints state may be heavy. Considerations such as this will affect how couplings run above M_G.

$$W_{symmetry\ breaking} =$$

$$A_1'(SA_1 + \mathcal{S}_1 A_1) + S'(\mathcal{S}_2 S + A_1^2)$$

$$+\tilde{A}'(\overline{\Psi}\Psi + \mathcal{S}_3 \tilde{A})$$

$$+A_2'(\mathcal{S}_4 A_2 + S\tilde{A} + (\mathcal{S}_1 + \mathcal{S}_5)\tilde{A})$$

$$+\overline{\Psi}' A_2 \Psi + \overline{\Psi} A_2 \Psi'$$

$$+A_1 A_2 \tilde{A}' + \mathcal{S}_6 (A_1')^2$$

17.3. Higgs sector

The Higgs sector is introduced below. It does not at the moment appear to be unique, but it is crucial for understanding the solution to several important problems – doublet-triplet splitting, μ problem and proton decay – and these constraints may only have one solution. The $10_1 A_1 10_2$ coupling is the term required by the Dimopoulos-Wilczek mechanism for doublet-triplet splitting. Since A_1 is an anti-symmetric tensor, we need at least two 10s.

The couplings of 10_1 to the 16s are introduced to solve the μ problem. After naturally solving the doublet-triplet splitting problem one has massless doublets. One needs however a small supersymmetric mass μ for the Higgs doublets of order the weak scale. This may be induced once SUSY is broken in several ways.

- The vev of the field A_1 may shift by an amount of order the weak scale due to the introduction of the soft SUSY breaking terms into the potential. In this theory the shift of A_1 appears to be too small.

- There may be higher dimension D terms in the theory of the form, eg.

$$\frac{1}{M_{Pl}} \int d^4\theta 10_1^2 (A_2^*).$$

Then supergravity effects might induce a non-vanishing vev for the F term of A_2 of order the $m_W M_G$. This will induce a value of μ of order $m_W M_G / M_{Pl}$. The shift in the F-terms also appear to be negligible.

- Higher dimension D-terms with hidden sector fields may however work. Consider $\frac{1}{M_{Pl}} \int d^4\theta 10_1^2 z^*$ where z is a hidden sector field which is connected with soft SUSY breaking. It would then be natural to have $F_z \approx \mu M_{Pl}$.

- One loop effects may induce a μ term once soft SUSY breaking terms are introduced[42]. In this case we find $\mu \sim \frac{A\lambda^4}{16\pi^2}$ where λ^4 represents the product of Yukawa couplings entering into the graph of figure 21.

  We use the last mechanism above for generating μ in the example which follows.

$$W_{Higgs} =$$

$$+\overline{\Psi}' A_2 \Psi + \overline{\Psi} A_2 \Psi'$$

$$+10_1 A_1 10_2 + \mathcal{S}_7 10_2^2$$

$$+\overline{\Psi}\overline{\Psi}' 10_1 + \Psi\Psi' 10_1$$

Note that the first two terms already appeared in the discussion of the symmetry breaking sector. They are included again here since as you will see they are important for the discussion of the Higgs sector as well. The last two terms are needed to incorporate the solution to the μ problem. As a result of these couplings to $\overline{\Psi}, \Psi$ the Higgs doublets in 10_1 mix with other states. The mass matrix for the SU(5) $\bar{\mathbf{5}}, \mathbf{5}$ states in $\mathbf{10_1}, \mathbf{10_2}, \mathbf{\Psi}, \mathbf{\Psi'}, \overline{\mathbf{\Psi}}, \overline{\mathbf{\Psi}}'$ is given below.

$$\begin{array}{c c}
 & \begin{array}{cccc} \bar{5}_1 & \bar{5}_2 & \bar{5}_\Psi & \bar{5}_{\Psi'} \end{array} \\
\begin{array}{c} 5_1 \\ 5_2 \\ 5_{\overline{\Psi}} \\ 5_{\overline{\Psi}'} \end{array} & \begin{pmatrix} 0 & A_1 & 0 & \Psi \\ A_1 & \mathcal{S}_7 & 0 & 0 \\ 0 & 0 & 0 & A_2 \\ \overline{\Psi} & 0 & A_2 & 0 \end{pmatrix}
\end{array}$$

Higgs doublets In the doublet sector the vev A_1 vanishes. Since the Higgs doublets in 10_1 now mix with other states, the boundary condition $\lambda_b/\lambda_t = 1$ is corrected at tree level. The ratio is now given in terms of a ratio of mixing angles.

Proton decay The rate for proton decay in this model is set by the quantity $(M^t)^{-1}_{11}$ where M^t is the color triplet Higgsino mass matrix[43]. We find $(M^t)^{-1}_{11} = \frac{\mathcal{S}_7}{A_1^2}$. This may be much smaller than $\frac{1}{M_G}$ for $\mathcal{S}_7$ sufficiently smaller than M_G. Note there

are no light color triplet states in this limit. Proton decay is suppressed since in this limit the color triplet Higgsinos in 10_1 become Dirac fermions (with mass of order M_G), *but they do not mix with each other.*

17.4. Symmetries

The theory has been constructed in order to have enough symmetry to restrict the allowed operators. This is necessary in order to reproduce the mass operators in the effective theory, as well as to preserve the vacuum directions assumed for the 45s and have natural doublet-triplet splitting. Indeed the construction of the symmetry breaking sector with the primed fields allows the 45s to carry nontrivial U(1) charges. This model has several unbroken U(1) symmetries which do not seem to allow any new mass operators. It has a discrete Z_4 R parity in which all the primed fields, $\mathcal{S}_{6,7}$ and 10_2 are odd and $16_i, i = 1, 2, 3$ and $\overline{\Psi}_a, \Psi_a, a = 1, \cdots, 9$ go into i times themselves. This guarantees that the odd states (and in particular, 10_2) do not couple into the fermion mass sector. There is in addition a Family Reflection Symmetry (see Dimopoulos- Georgi [14]) which guarantees that the lightest supersymmetric particle is stable. Finally, there is a continuous R symmetry which is useful for two reasons, (1) as a consequence, only dimension 4 operators appear in the superpotential and (2) this R symmetry is an anomalous Peccei-Quinn U(1) which naturally solves the strong CP problem.

Neutrino sector The neutrino sector seems to be very model dependent. It will constrain the symmetries of the theory, but I will not discuss it further here.

18. Conclusion

In this last lecture, I have presented a class of supersymmetric SO(10) GUTs which are in *quantitative* agreement with the low energy data. With improved data these particular models may eventually be ruled out. Nevertheless the approach of using low energy data to ascertain the dominant operator contributions at M_G seems robust. Taking it seriously, with *quantitative* fits to the data and including the leading order corrections to the zeroth order results, may eventually lead us to the correct theory.

What is the proverbial *smoking gun* for the theories presented here ? There are three observations which combined would confirm SUSY GUTs.

1. Gauge coupling unification consistent with the observed values of $\alpha, sin^2\theta_W, \alpha_s$.

2. Observation of SUSY particles.

3. Observation of proton decay into the modes $p \rightarrow K^+\overline{\nu}$ and $p \rightarrow K^0\mu^+$ [43]. Although SUSY GUTs may not predict the rate for this process, nevertheless the observation of this process would confirm SUSY GUTs.

In addition, the minimal SO(10) models presented here all demand large $\tan\beta$. Thus observation of large $\tan\beta$ would certainly strengthen these ideas. Finally, if the

calculable corrections to the predictions of one of these models improve the agreement with the data, it would be difficult not to accept this theory as a true description of nature.

19. Acknowledgements

I would like to thank all my collaborators with whom I have discussed much of the material in these lectures. I would also like to thank the organizers of the summer school for the enjoyable week I spent in Trieste. This work is supported in part by U.S. Department of Energy contract DE-ER-01545-640.

20. References

1. M. Shifman, N.G. Uraltsev and A. Vainshtein, "$|V_{cb}|$ from OPE Sum Rules for Heavy Flavor Transitions," University of Minnesota preprint, TPI-MINN-94/13-T, (LANL hep-ph/9405207) (1994).
2. C.T.H. Davies, K. Hornbostel, A. Langnau, G.P. Lepage, A. Lidsey, C.J. Morningstar, J. Shigemitsu, and J. Sloan, "A New Determination of M_b using LATTICE QCD," Ohio State preprint OHSTPY-HEP-T-94-004, (LANL hep-lat/9404012)(1994); for a review on α_s from the lattice, see Aida X. El-khadra, "The Strong Coupling from Quarkonia," Ohio State preprint OHSTPY-HEP-T-93-020, (LANL hep-lat/9312080)(1993). For recent results, see for example, C.R. Allton, M. Ciuchini, M. Crisafulli, E. Franco, V. Lubicz, and G. Martinelli, "Quark masses from lattice QCD at the next-to-leading order," Rome preprint 94/1018 (LANL hep-ph/9406263) (1994).
3. S. Weinberg, *Phys. Rev.*__D5__ 1962 (1972); H. Georgi and S. Glashow, *Phys. Rev.*__D7__ 2457 (1973).
4. G. 't Hooft, *Nucl. Phys.*__B33__ 173 (1971); G. 't Hooft, *Nucl. Phys.*__B35__ 167 (1971); G. 't Hooft and M. Veltman, *Nucl. Phys.*__B44__ 189 (1972).
5. S. Weinberg, *I.I. Rabi Festschrift* (1977); F. Wilczek and A. Zee, *Phys. Lett.*__70B__ 418 (1977); H. Fritzsch, *Phys. Lett.*__70B__ 436 (1977).
6. R. Gatto, G. Sartori and M. Tonin, *Phys. Lett.*__B28__ 128 (1968); N. Cabibbo and L. Maiani, *Phys. Lett.*__B28__ 131 (1968); R.J. Oakes, *Phys. Lett.* __B29__ 683 (1969).
7. C.D. Froggatt and H.B. Nielsen, *Nucl. Phys.*__B147__ 277 (1979). See also Z. G. Berezhiani, *Phys. Lett.* __129 B__ 99 (1983); ibid., __150 B__ 177 (1985); S. Dimopoulos, *Phys. Lett.*__129 B__ 417 (1983).
8. S. Dimopoulos and L. Susskind, *Nucl. Phys.*__B155__ 237 (1979); E. Eichten and K. Lane, *Phys. Lett.*__90B__ 125 (1980).
9. A. Masiero, D. Nanopoulos and Tamvakis, *Phys. Lett.*__126B__ 337 (1983); L.J. Hall, A. Kostelecky and S. Raby, *Nucl. Phys.*__B267__ 415 (1986); T. Banks, *Nucl. Phys.*__B303__ 172 (1988); R. Hemphling, DESY preprint 93-092 (1993); M. Dine, R. Leigh and A. Kagan, *Phys. Rev.*__D48__ 4269 (1993); Y. Nir and N.

Seiberg, *Phys. Lett.***B309** 337 (1993).
10. H. Fritzsch, *Nucl. Phys.***B155** 189 (1979). For a review, see F.J. Gilman and Y. Nir, *Ann. Rev. Nucl. Part. Sci.***40** 213 (1990).
11. M. Leurer, Y. Nir and N. Seiberg, *Nucl. Phys.***B398** 319 (1993); ibid., hep-ph/9310320 (1993).
12. H. Georgi and S. Glashow, *Phys. Rev. Lett.* **32** 438 (1974); J. Pati and A. Salam, *Phys. Rev.***D8** 1240 (1973).
13. H. Georgi, H. Quinn and S. Weinberg, *Phys. Rev. Lett.* **33** 451 (1974); S. Weinberg, *Phys. Lett.***91B** 51 (1980); C.H. Llewellyn-Smith, G.G. Ross and J.F. Wheater, *Nucl. Phys.***B177** 263 (1981); L.J. Hall, *Nucl. Phys.***B178** 75 (1981); P. Binetruy and T. Schucker, *Nucl. Phys.***B178** 293 (1981).
14. S. Dimopoulos, S. Raby and F. Wilczek, *Phys. Rev.* **D24** 1681 (1981); S. Dimopoulos and H. Georgi, *Nucl. Phys.***B 193** 150 (1981); L. Ibanez and G.G. Ross, *Phys. Lett.* **105B** 439 (1981); M. B. Einhorn and D. R. T. Jones, *Nucl. Phys.* **B196** 475 (1982); W. J. Marciano and G. Senjanovic, *Phys. Rev.* **D 25** 3092 (1982).
15. For recent analyses, see P. Langacker and N. Polonsky, *Phys. Rev.***D47** 4028 (1993); ibid., **D49** 1454 (1994); M. Carena, S. Pokorski, C.E.M. Wagner, *Nucl. Phys.***B406** 59 (1993).
16. P. Langacker and N. Polonsky, private communication (1994).
17. M. Chanowitz, J. Ellis and M.K. Gaillard, *Nucl. Phys.***B135** 66 (1978). For the corresponding SUSY analysis, see M. Einhorn and D.R.T. Jones, *Nucl. Phys.***B196** 475 (1982).
18. K. Inoue, A. Kakuto, H. Komatsu and S. Takeshita, *Prog. Theor. Phys.* **67** 1889 (1982); L.E. Ibañez and C. Lopez, *Nucl. Phys.***B233** 511 (1984).
19. H. Georgi and C. Jarlskog, *Phys. Lett.* **86B** 297 (1979).
20. H. Georgi and D.V. Nanopoulos, *Nucl. Phys.***B159** 16 (1979); J. Harvey, P. Ramond and D.B. Reiss, *Phys. Lett.***92B** 309 (1980); *Nucl. Phys.***B199** 223 (1982).
21. S. Dimopoulos, L.J. Hall and S. Raby, *Phys. Rev. Lett.***68** 1984 (1992);ibid., *Phys. Rev.***D45** 4192 (1992);ibid., **D46** R4793 (1992); P. Ramond, UFIFT-92-4 (1992); H. Arason, D. Castaño, E.J. Piard and P. Ramond, *Phys. Rev.***D47** 232 (1993); G. Anderson, S. Raby, S. Dimopoulos, and L. J. Hall, Phys. Rev. **D47** 3702 (1993).
22. V. Barger, M. S. Berger, T. Han and M. Zralek, Phys. Rev. Lett. **68** 3394 (1992); V. Barger, M. S. Berger and P. Ohmann, Phys. Rev. **D47** 1093 (1993);ibid., 2038 (1993).
23. K.S. Babu and R.N. Mohapatra, Bartol Research Institute preprint BA-94-56 (1994)(hep-ph/9410326).
24. K.S. Babu and R. Mohapatra, *Phys. Rev. Lett.***70** 2845 (1993); C. H. Albright and S. Nandi, FermiLab-conf. 94/094-T.
25. K.S. Babu and Q. Shafi, *Phys.Rev.***D47** 5004 (1993).
26. P. Ramond, R.G. Roberts and G.G. Ross, *Nucl. Phys.* **B406** 19 (1993).
27. F. Vissani and A. Smirnov, SISSA-63-94-EP (1994) (hep-ph/9405399) ; A.

Brignole, H. Murayama and R. Rattazzi, LBL-35774 (1994)(hep-ph/9406397).
28. B. Ananthanarayan, G. Lazarides and Q. Shafi, *Phys. Rev.***D44** 1613 (1991); H. Arason et al., *Phys. Rev. Lett.* **67** 2933 (1991); S. Kelley, J.Lopez and D. Nanopoulos, *Phys. Lett.* **B274** 387 (1992). See also the earlier work of M. Olechowski and S. Pokorski, *Phys. Lett.***B214** 393 (1988).
29. H. Georgi, "Lie Algebras in Particle Physics," Addison-Wesley Publishing Company (19).
30. G. Anderson, S. Dimopoulos, L.J. Hall, S. Raby and G. Starkman, *Phys. Rev.***D49** 3660 (1994).
31. S. Dimopoulos, *Phys. Lett.***129 B** 417 (1983); J. Bagger,S. Dimopoulos, H. Georgi and S. Raby, Proceedings of the Fifth Workshop on Grand Unification, Providence, Rhode Island, April 12 - 14, 1984, World Scientific Publishing Co., Singapore (1984).
32. S. Dimopoulos and F. Wilczek, Proceedings Erice Summer School, Ed. A. Zichichi (1981). For a recent discussion, see K.S. Babu and S.M. Barr, Delaware preprint BA-94-04 (1994).
33. D. Kaplan and A. Manohar, *Phys. Rev. Lett.***56** 2004 (1986); H. Leutwyler, *Nucl. Phys.***B337** 108 (1990).
34. R. Gupta, D. Daniel, G. W. Kilcup, A. Patel, and S. R. Sharpe, *Phys. Rev.***D47** 5113 (1993); S. Sharpe, in Lattice '91, Proceedings of the International Symposium, Tsukuba, Japan, 1991, edited by M. Fukugita et. al. [Nucl. Phys. B (Proc. Suppl.) **26** 197 (1992).]
35. Y. Nir and U. Sarid, *Phys. Rev.* **D47** 2818 (1993).
36. R. Rattazzi, U. Sarid and L.J. Hall, "Yukawa Unification: the Good, The Bad and The Ugly," Stanford preprint SU-ITP-94-15 (LANL hep-ph/9405313)(1994).
37. M. Carena, S. Pokorski, M. Olechowski, and C.E.M. Wagner, "Electroweak Symmetry Breaking and Bottom-Top Yukawa Unification," CERN Theoretical Report No. CERN-Th.7163/94 (LANL hep-ph/9402253)(1994).
38. L.E. Ibanez and G.G. Ross, *Phys. Lett.***110B** 215 (1982) ; L. Alvarez-Gaume, J. Polchinsky and M. Wise, *Nucl. Phys.***B221** 495 (1983) ; L.E. Ibanez and C. Lopez, *Phys. Lett.***126B** 54 (1983),ibid., *Nucl. Phys.***B233** 511 (1984) ; C. Kounnas, A.B. Lahanas, D.V. Nanopoulos and M. Quiros, *Nucl. Phys.***B236** 438 (1984); L.E. Ibanez, C. Lopez and C. Munoz, *Nucl. Phys.***B256** 218 (1985).
39. J. Ellis and M.K. Gaillard, *Phys. Lett.***88B** 315 (1979).
40. S. Chaudhuri, S. Chung and J.D. Lykken, FERMILAB PUB-94/137-T (LANL hep-ph/9405374)(1994); G. Cleaver, Talk given at Paschos 94; L. Ibañez, Talk given at DESY Workshop 94.
41. L.J. Hall and S. Raby, LBL-36357 (1995)(hep-ph/9501298).
42. L.J. Hall, in Supersymmetry and Supergravity/Nonperturbative QCD, Proceedings of the Winter School in Theoretical Physics, Mahabaleshwar, India, 1984, edited by P. Roy and V. Singh, Lecture Notes in Physics Vol. 208 (Springer, Berlin, 1984).
43. S. Dimopoulos, S. Raby and F. Wilczek, *Phys. Lett.***112B** 133 (1982); J.

Ellis, D.V. Nanopoulos and S. Rudaz, *Nucl. Phys.***B202** 43 (1982). For recent analyses, see R. Arnowitt and P. Nath, *Phys. Rev.***D49** 1479 (1994); J. Hisano, H. Murayama, and T. Yanagida, *Nucl. Phys.***B402** 46 (1993); P. Langacker, "Proton Decay," (LANL hep-ph/9210238) talk given at The Benjamin Franklin Symposium in Celebration of the Discovery of the Neutrino, Philadelphia, PA, 29 Apr - 1 May (1992).